# Arc Length and Riemannian Metric Geometry

**References:**


1. W. F. Reynolds, Hyperbolic geometry on a hyperboloid, *Amer. Math. Monthly* **100** (1993) 442–455.

2. Wikipedia page "Metric tensor". The most pertinent parts are at the beginning and end. (Beware that there are other, less relevant, pages with "metric" in their titles.)

3. Greenberg, Appendix A, Riemannian part.

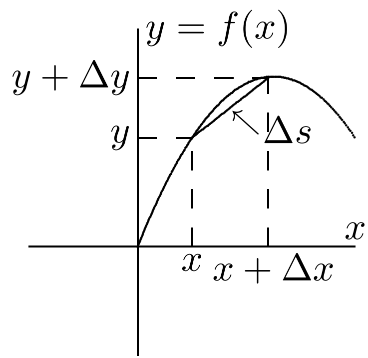
4. In the long run you might want to take Math. 439, "Differential Geometry of Curves and Surfaces", or Math. 460, "Tensors and General Relativity".


We shall look at a sequence of increasingly general or complicated situations.

## 1. Length of the graph of a function

The length of a short chord segment is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$



Drop the parentheses.

$$s \equiv \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \Delta s_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} = \int_{x_0}^{x_{\mathrm{f}}} \sqrt{1 + f'(x)^2}\, dx.$$

Shorthand: $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + f'(x)^2}\, dx.$

## 2. Length of a parametrized curve

Let $x = x(t)$, $y = y(t)$. $\Delta s^2 = \Delta x^2 + \Delta y^2$ still.

$$s = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \Delta t_i \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} = \int_{t_0}^{t_\mathrm{f}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Shorthand: $ds = \sqrt{dx^2 + dy^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$ (where $\dot{x} \equiv \frac{dx}{dt}$, etc.).

Special case: $t = x$. Then $\dot{x} = 1$, $ds = \sqrt{1 + \dot{y}^2}\, dx$ — same as situation 1.

## 3. Three dimensions

Let $x = x(t)$, $y = y(t)$, $z = z(t)$. Then

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad s = \int_{t_0}^{t_\mathrm{f}} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

## 4. Polar coordinates in the plane

Define $r$ and $\theta$ by $x = r\cos\theta$, $y = r\sin\theta$. A change $\Delta\theta$ changes lengths by $r\Delta\theta$. Therefore, it is intuitively clear (and we'll justify it in a moment) that

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Example: Consider the arc at $r = 2$ from $\theta = 0$ to $\theta = \pi/4$. Let $\theta$ play the role of the parameter, $t$.

$$\begin{aligned} ds &= \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + 2^2\left(\frac{d\theta}{d\theta}\right)^2}\, d\theta \\ &= \sqrt{0+4}\, d\theta = 2\, d\theta. \end{aligned}$$

$$s = \int_0^{\pi/4} 2\, d\theta = \frac{\pi}{2}.$$

In general,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt.$$

(Note that $r$ in the second term is a function of $t$, in general.)

A shorthand calculation leading to the arc length formula: Calculate

$$\begin{aligned} dx &= \cos\theta\, dr - r\sin\theta\, d\theta, \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta. \end{aligned}$$

(This really means that

$$\Delta x = \cos\theta\,\Delta r - r\sin\theta\,\Delta\theta + \text{something very small},$$

etc.) It follows that

$$\begin{aligned} dx^2 &= \cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 - 2r\sin\theta\cos\theta\, dr\, d\theta, \\ dy^2 &= \sin^2\theta\, dr^2 + r^2\cos^2\theta\, d\theta^2 + 2r\sin\theta\cos\theta\, dr\, d\theta. \end{aligned}$$

Therefore, $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

## 5. A curved surface embedded in 3-space

Example: The unit sphere, $x^2 + y^2 + z^2 = 1$. As usual let

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta.$$

Then, for instance, we have

$$dz = \cos\theta \, dr - r \sin\theta \, d\theta,$$

but $r$ is a constant in our problem ($r = 1$), so we can set $dr = 0$. Thus, ignoring $dr$, we have

$$dx = \cos\theta \cos\phi \, d\theta - \sin\theta \sin\phi \, d\phi,$$
$$dy = \cos\theta \sin\phi \, d\theta + \sin\theta \cos\phi \, d\phi.$$

It easily follows that

$$ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Note that near the "north pole", $\theta = 0$, we have $\sin\theta \approx \theta$ and hence $ds^2 \approx d\theta^2 + \theta^2\, d\phi^2$. That is, near the pole the angular spherical coordinates "look like" plane polar coordinates, with $\theta$ in the role of $r$ and $\phi$ in the role of $\theta$.

## 6. Indefinite metric (in dimension 2)

Consider a new definition of "length": Take the square of the distance from $(x_1, y_1)$ to $(x_2, y_2)$ to be

$$\Delta s^2 \equiv (x_1 - x_2)^2 - (y_1 - y_2)^2.$$

This quantity can be positive or negative (or zero). For today let's consider only separations for which it's positive, so we can take the square root and get a real number for the distance. In the limit of small distances we get the arc length formula

$$ds^2 = dx^2 - dy^2.$$

Example: Consider the hyperbola $y^2 - x^2 = R^2$, with $y > 0$ (so as to deal with only one branch). It can be parametrized by

$$x = R \sinh\theta \equiv R\,\frac{e^\theta - e^{-\theta}}{2}\,,$$
$$y = R \cosh\theta \equiv R\,\frac{e^\theta + e^{-\theta}}{2}\,.$$

Note or recall that $\cosh^2\theta - \sinh^2\theta = 1$. Now calculate

$$dx = R\cosh\theta\,d\theta, \qquad dy = R\sinh\theta\,d\theta.$$

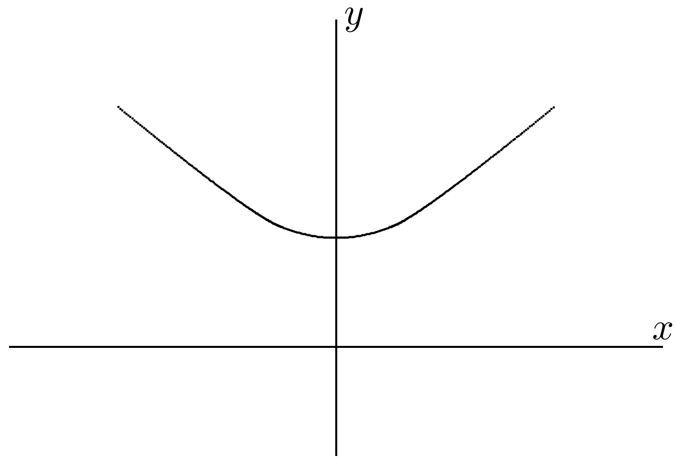
Therefore,

$$ds^2 = R^2(\cosh^2\theta - \sinh^2\theta)\, d\theta^2,$$

or

$$ds = R\, d\theta.$$

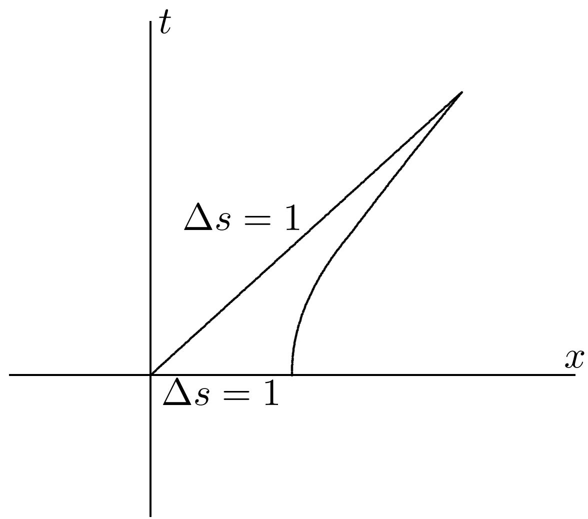
Remark: The geometry of space-*time* in special relativity is like this. The $y$ coordinate is interpreted as time. Therefore, I shall henceforth write it as $t$.




Diagrams can be misleading, because the lengths in this geometry are not the same as the Euclidean lengths of the drawn segments representing them. For example, the hyperbola $x^2 - t^2 = 1$ can be parametrized by

$$x = \cosh\alpha, \quad t = \sinh\alpha.$$

All points on this curve are the same "distance" (namely, 1) from the origin $(0, 0)$ with respect to the indefinite metric $ds$, but their Euclidean distances, $\sqrt{\cosh^2 \alpha + \sinh^2 \alpha}$, become arbitrarily large.



## 7. Indefinite metric in dimension 3, and a surface embedded there

This time we have coordinates $x$, $y$, and $t$, and the metric (arc length formula)

$$ds^2 = dx^2 + dy^2 - dt^2.$$

(Here $t$ plays the role pioneered by $y$ in the previous situation, and $y$ is now just a second ordinary spatial coordinate.)

Consider the top half ($t > 0$) of the hyperbolic surface $x^2 + y^2 - t^2 = -R^2$. It can be parametrized by

$$t = R\cosh r, \qquad x = R\sinh r\cos\theta, \qquad y = R\sinh r\sin\theta.$$

Then

$$dt = R\sinh r\, dr,$$

$$dx = R\cosh r\ \cos\theta\,dr - R\sinh r\sin\theta\,d\theta,$$

$$dy = R\cosh r\ \sin\theta\,dr + R\sinh r\cos\theta\,d\theta.$$

Thus $ds^2 = R^2\left(dr^2 + \sinh^2 r\,d\theta^2\right)$. Recall that this means that the length of a curve $\mathbf{r}(\tau)$ is

$$\int_{\tau_0}^{\tau_{\mathrm{f}}} R\sqrt{\left(\frac{dr}{d\tau}\right)^2 + \sinh^2\big(r(\tau)\big)\left(\frac{d\theta}{d\tau}\right)^2}\,d\tau.$$

This surface will prove to be a very important model of hyperbolic geometry.

*Geodesics* are curves that minimize length between fixed neighboring points. Their parametrizations $\mathbf{r}(\tau)$ satisfy a certain differential equation. But luckily they have a more elementary characterization in our case: intersections of the hyperboloid with planes through the origin in the 3-dimensional space (Fig. 7.19). These are the analogs of the great circles on a sphere, and hence of straight lines in a Euclidean plane.

Note that the “radius” $R$ is a scale factor distinguishing different models of hyperbolic (or spherical) geometry, with Euclidean plane geometry emerging in the limit $R \to \infty$.