Math. 467 (Fulling) 14 December 2015

Final Examination – Solutions

Name: ________________________________

1. *(Multiple choice – each 5 pts.)* Circle the correct capital letter.

   (a) *Similarity* of triangles
   - (A) is not an equivalence relation.
   - (B) is an equivalence relation, and the cells of the partition it defines are larger than those of the congruence relation.
   - (C) is an equivalence relation, and the cells of the partition it defines are the same as those of the congruence relation.
   - (D) is an equivalence relation, and the cells of the partition it defines are smaller than those of the congruence relation.
   - (E) is an equivalence relation, and the cells of the partition it defines have no simple relation to those of the congruence relation.

B

   (b) The **converse** of the alternate interior angle theorem
   - (A) is equivalent (in Hilbert's neutral geometry) to the Euclidean parallel postulate.
   - (B) was first proved by Legendre.
   - (C) shows that the elliptic parallelism property is inconsistent with Hilbert's axioms.
   - (D) is satisfied by Dehn's models, because they have very few parallel lines.
   - (E) shows that Saccheri's "hypothesis of the obtuse angle" is never true.

A

   (c) A model of 2-dimensional hyperbolic geometry that can be realized by a physical object in 3-dimensional Euclidean space is
   - (A) the Klein disk.
   - (B) the Poincaré disk.
   - (C) the Poincaré half-plane.
   - (D) the Beltrami pseudosphere
   - (E) the hyperboloid.

D

   (d) In the real hyperbolic plane
   - (A) the angles of a triangle sum to less than 180°.
   - (B) the angle sum of a triangle increases as the area of the triangle increases.
   - (C) both the alternate interior angle theorem and its converse are true.
   - (D) the circumference of a circle is less than π times the diameter.
   - (E) [all of these].

A [B and D are elliptic properties.]

(e) Today we know that a proof of the parallel postulate from Euclid's other axioms
  (A) is still a very important unsolved problem.
  (B) is possible only in Dehn's models and similar exotic cases.
  (C) is impossible, because it would show that mathematics as we know it is inconsistent.
  (D) is possible but extremely unlikely, because it would show that mathematics as we know it is inconsistent.

D

(f) Axiom C-4 authorizes us to construct an angle $\angle \mathrm{B}'\mathrm{A}'\mathrm{C}'$ congruent to a given angle $\angle \mathrm{BAC}$ adjacent to a given ray $\overrightarrow{\mathrm{A}'\mathrm{B}'}$. The result is unique provided that
  (A) the *side* of the line $\overleftrightarrow{\mathrm{A}'\mathrm{B}'}$ on which the new angle appears is prescribed.
  (B) segment AB is congruent to segment $\mathrm{A}'\mathrm{B}'$.
  (C) the Euclidean parallel postulate holds.
  (D) $\angle \mathrm{BAC}$ is a right angle.
  (E) congruence of angles is an equivalence relation.

A

(g) In a real hyperbolic plane, consider a line m through a point P and parallel to a line l. Which of these is true?
  (A) Lines m and l are equidistant.
  (B) There is always a unique line between m and l that is perpendicular to both.
  (C) If it exists, the unique perpendicular between m and l maximizes the distance between them.
  (D) If m is a limiting parallel ray from P to l, then there is no line perpendicular to both l and m.
  (E) There is exactly one limiting parallel ray from P to l.

D (See Greenberg Fig. 6.14 and surrounding text.)

(h) Let P stand for the proposition "All right angles are congruent" and let Q stand for the SAS triangle congruence criterion. Which of the following is true?
  (A) Both P and Q were assumed as axioms by Euclid.
  (B) Both P and Q were proved as theorems by Euclid.
  (C) Q can easily be proved from P.
  (D) Modern (Hilbert) geometry has Q as an axiom and P as a theorem, but in Euclid's writing it was the reverse.
  (E) Modern (Hilbert) geometry has P as an axiom and Q as a theorem, but in Euclid's writing it was the reverse.

D

2. *(10 pts.)* Lengths of curves in the Poincaré half-plane are computed from the formula

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$

Find the Poincaré length of the segment AB if A has coordinates $(2, 1)$ and B has coordinates $(2, 3)$.

This segment is part of the Poincaré line $x = 2$. Since x is constant, we may write the arc-length formula as $ds = \sqrt{dy^2/y^2} = dy/y$. Therefore, the length is

$$s = \int_1^3 \frac{dy}{y}\,dy = \ln y\Big|_1^3 = \ln 3\,.$$

3. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

Bolyai, Euclid, Hilbert, Saccheri, Pythagoras, Proclus

Pythagoras, Euclid, Proclus, Saccheri, Bolyai, Hilbert

4. *(15 pts.)*

(a) Define *Saccheri quadrilateral.*

[See Greenberg pp. 176–177.]

(b) Prove that the summit angles (which you should also define) of a Saccheri quadrilateral are congruent.

[See Greenberg pp. 177–178.]

5. *(15 pts.)* Prove the SSS criterion for triangle congruence: If AB $\cong$ DE, BC $\cong$ EF, and AC $\cong$ DF, then $\triangle$ABC $\cong$ $\triangle$DEF.

[This is Greenberg Exercise 3.32, which we discussed extensively in class as well as homework.]

6. *(Essay – 40 pts.)* Carefully verify that the Klein disk model satisfies all the Hilbert *incidence* and *betweenness* axioms and the *hyperbolic parallel property* (i.e., existence of multiple parallels to a given line through a given point). This is 8 statements in all. You may assume that the underlying Euclidean plane is the standard ("real") Euclidean plane, which satisfies Dedekind's axiom.

"Carefully" means that you can't just say "This is obvious." Typically, you will need to cite a Euclidean fact (which you don't need to reprove) and then explain why it is still applicable. (For example, you might need to check that a point whose existence is guaranteed by a Euclidean axiom does not fall outside the disk.) In some cases, a drawing will help.