

Final Examination – Solutions

1. (*Multiple choice – each 5 pts.*)

- (a) The curvature of a surface is
- (A) negative for a sphere and positive for a hyperboloid.
 - (B) negative when circumferences are smaller than 2π times radii and positive when angle sums (of triangles) are greater than 180° .
 - (C) positive when circumferences are smaller than 2π times radii and negative when angle sums are less than 180° .
 - (D) positive when circumferences are smaller than 2π times radii and negative when angle sums are greater than 180° .

C

(b) The arc-length formula in the **Poincaré half-plane** is

- (A) $ds^2 = (dx^2 + dy^2)/y^2$
- (B) $ds^2 = dr^2 + r^2 d\theta^2$
- (C) $ds^2 = dr^2 + \sin^2 r d\theta^2$
- (D) $ds^2 = dr^2 + \sinh^2 r d\theta^2$

A

(c) Which of these is **not equivalent** to the Hilbert parallel postulate, HE?

- (A) the converse of the alternate interior angle theorem
- (B) the triangle inequality
- (C) Euclid's original Postulate V
- (D) Given a triangle, there exists a similar triangle of any desired size.

B

(d) Axioms to replace “Euclid V” were proposed by all of these **except**

- (A) Archimedes
- (B) Proclus
- (C) Wallis
- (D) Clavius

A (Actually, I do not know for certain that Archimedes made no such proposal, but I have read dozens of 4-page biographies of Archimedes that did not mention such a thing.)

(e) Dehn's models show that

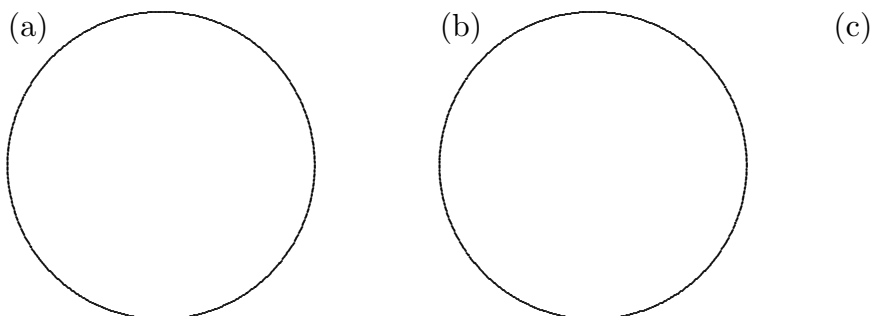
- (A) geometries that do not satisfy Dedekind's axiom are inconsistent.
- (B) if the sum of the angles in a triangle is always 180° , then Euclid's parallel postulate holds.
- (C) elliptic parallelism is consistent with the Hilbert IBC axioms.
- (D) hyperbolic parallelism is not exactly equivalent to the acute angle hypothesis.

D

- (f) Numerical measurement of angles and segments (e.g., degrees and meters)
- (A) is impossible in Euclidean geometry, because of Zeno's paradoxes.
 - (B) is possible only if the parallel postulate is assumed.
 - (C) is contrary to the spirit of Euclid's approach, but nevertheless is perfectly rigorous from a modern point of view.
 - (D) became possible only after Hilbert introduced the real numbers into geometry.

C

2. (21 pts.) "Legendre's axiom" states that every point in the interior of an angle lies on a segment joining a point on one side of the angle to a point on the other side. This claim is **false** in hyperbolic geometry. Draw sketches showing how it can fail, in (a) the Klein disk picture, (b) the Poincaré disk picture, and (c) the naive picture we assumed in Chapter 3, where the angle is drawn in a plane. For your convenience, disks are provided for (a) and (b).



[In (a) the 3 lines should be straight. In (b) they should be circular arcs (which may be straight if they pass through the disk's center). In (c) the line of enclosure must be curved and should be asymptotic to the two sides of the angle. In each case, to make the point there should be a point outside the triangle formed.]

3. (20 pts.)

(a) State the **exterior angle theorem** (the version that is a theorem in neutral geometry).

[See p. 164.]

(b) State the **stronger** version of the exterior angle theorem that is true in Euclidean geometry.

An exterior angle is congruent to the sum of the two remote interior angles (p. 176).

(c) Each of those two theorems is equivalent (respectively) to a certain theorem about a sum of **interior** angles of an arbitrary triangle. State those two angle-sum theorems.

For (a), "The sum of the degree measures of any two angles of a triangle is less than 180° ." (p. 171)

For (b), "The sum of the angles of a triangle is 180° ." (p. 175).

4. (30 pts.) Rearrange these names into historical order, earliest to latest:

Pythagoras, Lambert, Euclid, Lobachevsky, Klein, Proclus

Pythagoras, Euclid, Proclus, Lambert, Lobachevsky, Klein

5. (24 pts.) Prove **ONE** of these. **If you try both, clearly indicate which one you want graded.**

- (A) **AIA:** If two lines have a pair of congruent alternate interior angles with respect to a transversal, then those two lines are parallel.
- (B) In any bi-right quadrilateral, the greater side is opposite the greater summit angle. (Assume known that the summit angles of a Saccheri quadrilateral are congruent.)

6. (Essay – 25 pts.)

This course has centered on the Euclidean parallel postulate. Your essay should explain what the postulate is (concentrate on the Proclus–Playfair form), explain its logical status (proved? refuted? whatever?), and discuss the historical and philosophical significance of this issue.