Math. 467 (Fulling) 13 October 2017

Midterm Test – Solutions

1. *(Multiple choice – each 5 pts.) (Circle the correct capital letter.)*

   (a) Euclid's geometry is based on five fundamental, undefined terms, but the importance of one of them was not fully realized until the work of Pasch and Hilbert over 2000 years later. Which one?

   (A) point
   (B) line
   (C) lie on
   (D) between
   (E) congruent

D

   (b) Which of these is **not** a theorem of Hilbert geometry (without a parallel postulate)?

   (A) If $\angle\mathrm{CAB} < \angle\mathrm{DEF}$ and $\angle\mathrm{DEF} < \angle\mathrm{GHI}$, then $\angle\mathrm{CAB} < \angle\mathrm{GHI}$.
   (B) If D is in the interior of $\angle\mathrm{CAB}$, then so is every other point on $\overrightarrow{\mathrm{AD}}$ except A.
   (C) If $\angle\mathrm{CAB} < \angle\mathrm{DEF}$ and $\angle\mathrm{DEF} \cong \angle\mathrm{GHI}$, then $\angle\mathrm{CAB} < \angle\mathrm{GHI}$.
   (D) If $\overrightarrow{\mathrm{AD}}$ is between $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AB}}$, then $\overrightarrow{\mathrm{AD}}$ intersects segment BC.
   (E) Every point D in the interior of $\angle\mathrm{CAB}$ lies on a segment joining a point E on $\overrightarrow{\mathrm{AB}}$ to a point F on $\overrightarrow{\mathrm{AC}}$.

E (This is the subject of the "Warning" on p. 115.)

   (c) Simplify $\neg\forall x\,\exists y\,\big[[(x<y)\wedge\exists z\,(x+z>y)] \;\Rightarrow\; (x+y\le 3)\big]$.

   (A) $\exists x\,\forall y\,\big[(x<y)\wedge\exists z\,(x+z>y) \;\Rightarrow\; (x+y>3)\big]$.
   (B) $\exists x\,\forall y\,\big[(x<y)\wedge\exists z\,(x+z\le y)\wedge(x+y>3)\big]$.
   (C) $\exists x\,\forall y\,\big[\big((x<y)\wedge\exists z\,(x+z>y)\big)\vee(x+y>3)\big]$.
   (D) $\exists x\,\forall y\,\big[\big((x\ge y)\vee\forall z\,(x+z\le y)\big)\wedge(x+y>3)\big]$.
   (E) $\exists x\,\forall y\,\big[(x<y)\wedge\exists z\,(x+z>y)\wedge(x+y>3)\big]$.

E

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2. *(12 pts.)* State the *crossbar theorem* and draw a sketch to illustrate it.

[See p. 116.]

3. *(15 pts.)* Present a model of the three incidence axioms, containing only finitely many points, that satisfies the *Euclidean* parallelism property. How much freedom do you have in choosing the number of points?

Take 4 points, and let the lines be the 2-point subsets. Four is the only number that gives a Euclidean result.

4. *(18 pts.)* List the three congruence axioms that involve angles. (These are the last three.)

[See pp. 120–121 .]

5. *(20 pts.)* Do **ONE** of these. *(Extra credit for doing both is limited to 10 points.)*

   (A) State and prove the SSS triangle congruence theorem.

   (B) Prove the corollary to Axiom B-4:

   If A and B are on opposite sides of l and if B and C are on the same side of l, then A and C are on opposite sides of l.

6. *(Essay – 20 pts.)* Let $\mathcal{S}$ be a set. Recall that an *equivalence relation* on $\mathcal{S}$ is a binary relation $\sim$ with the properties (for all A, B, C in $\mathcal{S}$)
   (1) reflexivity: $A \sim A$,
   (2) symmetry: $A \sim B \;\Rightarrow\; B \sim A$,
   (3) transitivity: $A \sim B \wedge B \sim C \;\Rightarrow\; A \sim C$.
   Recall also that a *partition* of $\mathcal{S}$ is a collection $\{S_i\}$ of subsets of S with the properties
   (4) disjointness: $S_i \cap S_j = \emptyset$ if $i \neq j$,
   (5) exhaustiveness: $S = \bigcup_i S_i$.

   Show that, given a partition of $\mathcal{S}$, the definition

$$A \sim B \iff A \text{ and } B \text{ belong to the same } S_i$$

   defines an equivalence relation on $\mathcal{S}$. Then show that, conversely, every equivalence relation on $\mathcal{S}$ determines a partition of $\mathcal{S}$.