Math. 467 (Fulling) 11 May 2009

Final Examination

Name: ______________________________

1. *(40 pts.)* was the take-home essay.

2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

   Saccheri, Euclid, Bolyai (the son), Klein, Thales, Proclus

3. *(30 pts.)* State and prove the *Alternate Interior Angle Theorem* (AIA). *Hint:* This theorem shows that spherical and elliptic geometries are inconsistent with Hilbert's axioms, so the proof must make use (at least indirectly) of Axioms I-1 and B-4.

4. *(30 pts.)* Do **ONE** of these: (Continue on next page if necessary.)
(NO extra credit for doing both. Indicate which one you want graded!)

(A) In the Poincaré disk model, the formula for arc length is

$$ds^2 = \frac{4(dx^2 + dy^2)}{[1 - (x^2 + y^2)]^2} \,.$$

Introduce polar coordinates (ρ, θ) in the (x, y) plane and show that the further coordinate transformation

$$\rho = \tanh\left(\frac{r}{2}\right)$$

converts the arc length to

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2 \,.$$

Explain why this result justifies the vague claim that hyperbolic geometry describes "a sphere of imaginary radius".

(B) Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The *cross-ratio* is defined as

$$(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}} \, \frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}} \,,$$

where, for instance, $\overline{\mathrm{AB}}$ is the *Euclidean* distance between A and B. Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is *additive* in the sense that

$$d(\mathrm{AB}) = d(\mathrm{AC}) + d(\mathrm{CB})$$

when $\mathrm{A} * \mathrm{C} * \mathrm{B}$ along a Poincaré line. (Verify that it is indeed additive.)

5. *(Essay – 30 pts.)* Define the *defect* of a triangle; explain how it differs among Euclidean, hyperbolic, and elliptic geometries; add any other related historical and mathematical facts that you recall.

6. *(40 pts.)* Pronounce each of the following assertions true or false. **To avoid ambiguity, please WRITE OUT the word "true" or "false".**

   (a) Although 2000 years of efforts to prove the parallel postulate as a theorem in neutral geometry have been unsuccessful, it is still possible that someday some genius will succeed in proving it.

   (b) In the Poincaré disk model, two Poincaré lines are considered "perpendicular" if and only if they are perpendicular in the usual Euclidean sense.

   (c) In hyperbolic geometry, the summit angles of Saccheri quadrilaterals are always acute.

   (d) In hyperbolic geometry, any two parallel lines have a common perpendicular.

   (e) Every valid theorem of neutral geometry is also valid in hyperbolic geometry.

   (f) If we add to the axioms of neutral geometry the postulate that *no* parallel lines exist), we get another consistent geometry in addition to the Euclidean and hyperbolic ones.

   (g) It is a theorem in neutral geometry that if $l \parallel m$ and $m \parallel n$, then $l \parallel n$.

   (h) It is a theorem in neutral geometry that if l and m are parallel lines, then alternate interior angles cut out by any transversal to l and m are congruent to each other.

   (i) It is impossible to prove in neutral geometry that rectangles exist.

   (j) The SSS criterion for congruence of triangles is a theorem in neutral geometry.