Chapter 3

Betweenness (ordering)

"Point B is *between* point A and point C" is a fundamental, undefined concept. It is abbreviated $A * B * C$.

A system satisfying the incidence and betweenness axioms is an *ordered incidence plane* (p. 118).

The first 3 axioms (p. 108): [Refer to the Pitts summary.]

B-1 is really three statements.

Corollary to B-2 and B-3: There are infinitely many points.

Note to B-3: The real projective plane is excluded. (This is our first hint that elliptic geometries do not fit well with the Hilbert axioms. In Ch. 4, p. 163, we will prove that parallel lines always exist, so the elliptic parallelism property is not consistent with the Hilbert axioms.)

Definitions and notations (same as in Ch. 1; p. 109)**:**

- [*line* $\overleftrightarrow{\mathrm{AB}}$ is an undefined concept.]
- *segment* AB (The segment includes the endpoints, though the betweenness relation $A * C * B$ does not.)
- *ray* $\overrightarrow{\mathrm{AB}}$
- set of points on a line, $\{\overleftrightarrow{\mathrm{AB}}\}$

Proposition 3.1. [Proof of (i) in book, p. 109; proof of (ii) by a team next time]

Correction to p. 110: "Exercise 17" should be "Exercise 16."

Definitions: *same side* and *opposite side* (p. 110). Let's introduce some abbreviations: (It is always understood that A and B do not lie on the given line l.)

- (A, B) same *means* Either $A = B$ or segment AB does not intersect l.
- (A, B) opposite *means* $A \neq B$ and AB does intersect l.

Note that

- The language is loaded: What is a "side"?

- These relations are *symmetric*:

$$(B, A) \text{ same} \iff (A, B) \text{ same},$$

  and similarly for "opposite".

- The hypotheses of the two parts exhaust all possibilities. Thus

$$(A, B) \text{ opposite} \iff \neg(A, B) \text{ same} \tag{$*$}$$

  (always with the understanding that neither A nor B lies on the line itself). (Greenberg does not point out $(*)$ explicitly, but mentions "excluded middle" every time it is used.)

Axiom B-4 (plane separation property) and corollary (pp. 110–111):

(i) $$(A, B) \text{ same and } (B, C) \text{ same} \Rightarrow (A, C) \text{ same}.$$

(ii) (A, B) opposite and (B, C) opposite $\Rightarrow$ (A, C) same.

(iii) (A, B) opposite and (B, C) same $\Rightarrow$ (A, C) opposite.

Corollary (iii) follows from (i) and $(*)$ and the fact that "same side" is a symmetric relation: Note these tautologies:

$$(q \Rightarrow r) \iff (\neg r \Rightarrow \neg q).$$
$$(q \wedge p \Rightarrow r) \iff [p \Rightarrow (q \Rightarrow r)].$$

Thus (i) can be rewritten as

$$(B, C) \text{ same } \Rightarrow [(A, B) \text{ same } \Rightarrow (A, C) \text{ same}],$$

or

$$(B, C) \text{ same } \Rightarrow [(A, C) \text{ opposite } \Rightarrow (A, B) \text{ opposite}],$$

hence

$$(A, C) \text{ opposite and } (B, C) \text{ same } \Rightarrow (A, B) \text{ opposite.}$$

Since B and C are arbitrary letters, they can be interchanged, and the result is exactly (iii).

Definition: *side* = *half-plane* (p. 111)

The situation can now be summarized by saying that "being on the same side" is an *equivalence relation*, or, equivalently, that the sides of a line form a *partition* of the plane (more precisely, of the points in the plane that don't lie on the line). Furthermore, because of (ii) there are only two sides. These observations make up **Proposition 3.2**.

Note to B-4: Dimensions higher than 2 are now excluded.

Side discussion of equivalence relations (cf. pp. 82–83)

- *equivalence relation:* A (binary) relation $\sim$ is an equivalence relation if it has these 3 famous properties:

  * *reflexive:* $A \sim A$ (for all A in some universe S).

  * *symmetric:* $A \sim B \;\Rightarrow\; B \sim A$

  * *transitive:* $A \sim B \wedge B \sim C \;\Rightarrow\; A \sim C$

- *equivalence classes:* $[A] \equiv \{B \colon B \sim A\}$

- *partition:* A collection $\{S_i\}$ of subsets of a set S form a partition of S if:

  * The subsets are (pairwise) disjoint: $S_i \cap S_j = \emptyset$ if $i \neq j$.

* The subsets exhaust S: $S = \bigcup_i S_i$.

The sets S_i are called “cells” of the partition.

Proposition: Given an equivalence relation on S, its equivalence classes form a partition of S. Conversely, given a partition, the condition “A and B belong to the same cell S_i” defines an equivalence relation.

Remark: In an algebraic context equivalence classes are often called *cosets.* For example, lines and planes in Euclidean geometry (affine subspaces) are cosets of the underlying linear algebra, the equivalence relation on the vectors being that their difference belongs to the true subspace (line or plane through the origin) that is parallel to the affine subspace in question.

- *factor space:* Think of each equivalence class (or partition cell) as a single point in a new space (smaller but more abstract). Many interesting spaces are constructed this way. An example we recently saw is the *real projective plane*, whose points were defined to be lines (sets of points related by scaling).

Back to betweenness . . .

Proposition 3.3 (pp. 112–113) [prove first part; rest for a team]

Corollary [team]

[Henceforth repeat book theorems only when necessary to filibuster for teams.]

Proposition 3.4 (line separation property) (p. 113–114)

Pasch's Theorem (p. 114)

Proposition 3.5 [team]

Proposition 3.6 [team]

Exercise 9 (useful for proving Props. 3.7 and 3.8)**:** Given a line l, a point A on l, and a point B not on l; then every point of the ray $\overrightarrow{\mathrm{AB}}$ (except A) is on the same side of l as B.

Lemma: If $\mathrm{C} \in \overrightarrow{\mathrm{AB}}$ and $\mathrm{C} \neq B$, then $\mathrm{A} \ \mathtt{I} \ \overleftrightarrow{\mathrm{BC}}$.

Proof of Lemma: C on the ray $\Rightarrow$ C, A, and B are collinear $\Rightarrow$ A on the line.

Proof of Exercise: Assume $\mathrm{C} \in \overrightarrow{\mathrm{AB}}$ but C and B are on opposite sides of l. Then BC intersects l, and by the lemma and Prop. 2.1, that intersection point must be A. Therefore, C*A*B. However, by definition of the ray, either $\mathrm{C} \in \mathrm{AB}$

or A*B*C; the first of these possibilities expands to either A*C*B or C = A (excluded by hypothesis) or C = B (contradicting "opposite sides"). So we have a contradiction with B-3. Therefore, C and B must be on the same side.

Definition: *interior* for angles (p. 115)

Proposition 3.7 [team]

Proposition 3.8 [team]

Definition: *betweenness* for rays, and the warning above it (p. 115)

Crossbar Theorem (p. 116)

Definitions: *interior* and *exterior* for triangles (p. 117)

Proposition 3.9 [team]

Congruence

Congruence is an undefined term for both segments and angles, and a defined term for triangles.

Axiom C-1 (p. 119) is essentially "Euclid II" (p. 16).

Axioms C-2 and C-5: Congruence of segments and congruence of angles are both equivalence relations.

Remark: Greenberg assumes "twisted transitivity"

$$a \cong b \wedge a \cong c \;\Rightarrow\; b \cong c$$

and reflexivity and proves symmetry. Ordinary, but not twisted, transitivity is satisfied by order relations such as $\leq$.

Axiom C-3: "Equals added to equals" for segments along their respective lines. (The analog for angles will be a theorem (Prop. 3.19). Likewise the analogs for subtraction, Prop. 3.11 and 3.20.)

Axiom C-4: A given angle can be attached to a given ray on either side (but otherwise uniquely).

Axiom C-6: SAS congruence criterion. This famous "theorem" turns out to be independent of the other axioms. (We will prove that in Exercise 35.) Euclid's "proof" uses an intuitive concept of moving figures rigidly that is foreign to his axioms and standard methods.

Proposition 3.10: (p. 123) An isosceles triangle has equal base angles. (Proof in book. Converse in homework, Prop. 3.18. The points in Fig. 3.20 on p. 123 are mislabeled.) Note that the Pappus proof and the definition of congruence of triangles regard reflection-symmetric triangles as congruent. Recall that in the definition of “angle” on p. 18 we have a *set* of two rays, not an *ordered pair* of rays, so an angle is identical with itself “in reverse”.

Proposition 3.12 and **Definitions** of segment ordering (p. 124): See book and use in the following.

Proposition 3.13: (segment ordering, p. 125) Work out in class: Exercises 21–23.

For the next batch of propositions (p. 125) we need to import some definitions from Chapter 1:

supplementary angles

vertical angles (Exercise 1.4)

right angles

perpendicular lines (The passage from rays to lines (some tedious logical bookkeeping) is stated in the exercise section, p. 42.)

Proposition 3.14: Supplements of congruent angles are congruent. (proved in exercise section)

Proposition 3.15: [prove in lecture]

(a) Vertical angles are congruent to each other.

(b) An angle congruent to a right angle is a right angle.

Proposition 3.16: For every line l and every point P there is a line through P perpendicular to l. [proved in book; go through it] Uniqueness is not yet clear.

Propositions 3.17 and 3.22: ASA and SSS. (homework)

[Discuss proof of SSS if time permits.]

Propositions 3.19–21 deliver *angle* addition, subtraction, and ordering, in analogy with segments. (The last is a major part of the homework.)

Proposition 3.23: (p. 128) "Euclid IV" — All right angles are congruent. [go through proof if time permits]

Ordering for angles (and Euclid IV) allows *acute* and *obtuse* to be defined.

A *Hilbert plane* is a model satisfying all the I, B, and C axioms.

[Work out Exercise 35 (congruence part) to show independence of SAS. Prove SSS if time permits.]

Continuity

Greenberg states a large number of rival continuity axioms, but he prefers two:

Circle-Circle Continuity Principle: If circle γ has one point inside and one point outside circle γ', then the two circles intersect in two points.

Dedekind's Axiom: If $\{l\}$ is a disjoint union $\Sigma_1 \cup \Sigma_2$ with no point of one subset between two points of the other (and neither subset empty), then there exists a unique O on l such that either Σ_1 or Σ_2 is a ray of l with vertex O (and then the other subset is the opposite ray with O omitted). (I.e., each line is (at least locally) like the real line $\mathbf{R}$ as Dedekind defined the latter. "At least locally" means that the geometrical line may be modeled by an interval of $\mathbf{R}$, not all of $\mathbf{R}$.)

Dedekind's assumption is very strong; circle-circle is fairly weak (and implied by Dedekind). Some kind of continuity assumption is necessary to guarantee that the would-be intersection points of circles with lines, segments, or other circles are not accidentally "holes" in the space. For example, the *rational* plane $\mathbf{Q}^2$ lacks points with irrational coordinates, so "$y = \sqrt{1 - x^2}$" may be problematical. (In fact, $\mathbf{Q}^2$ has an even worse problem, which we'll see on some later day.)

We do not commit to any one continuity axiom.

A recent paper by Greenberg, *Amer. Math. Monthly* **117** (2010) 198–219, reports some more recent and/or more advanced theorems on the relations among various continuity axioms (among other things).

Hilbertian parallelism

Hilbert's Euclidean Axiom of Parallelism: For every l and every P not on l, there is *at most* one line though P and parallel to l.

This seemingly leaves open the possibility that there are *no* parallels for the given l and P. (Recall that if there are *never* parallels, the geometry would be classed as elliptic.) However, it turns out that the other Hilbert axioms already rule out this possibility: Hilbert geometries are either Euclidean or hyperbolic. (An axiomatic development of elliptic geometry requires different axioms; see Appendix A.)

Definitions: A *Euclidean plane* is a Hilbert plane satisfying Hilbert parallelism and circle-circle continuity. A *real Euclidean plane* is a Hilbert plane satisfying Hilbert parallelism and Dedekind continuity.

Not surprisingly, $\mathbf{R}^2$ is a real Euclidean plane (essentially the only one). As mentioned, $\mathbf{Q}^2$ is not even a (circle-circle) Euclidean plane. To get one, we need to enlarge $\mathbf{Q}$ to contain enough algebraic numbers to take square roots. A non-Dedekind Euclidean plane is described on pp. 140–142. It is $\mathbf{K}^2$, where $\mathbf{K}$ is the field of *constructible* numbers, which includes all the square roots of rational numbers, and all the square roots of other constructible numbers. The name goes back to the problem of constructing — with compass and straightedge — a segment of a given length, relative to another segment.