## Final Examination - Solutions

1. (Multiple choice - each 5 pts.)
(a) Lambert quadrilaterals
(A) were shown to be impossible by Clairaut.
(B) have exactly 3 right angles unless at least one rectangle exists.
(C) can be divided into two Saccheri rectangles.
(D) have an angle sum of $360^{\circ}$ in the Poincaré disk model.

B
(b) A proof of Euclid's parallel postulate from the Hilbert IBC axioms would establish that
(A) the acute angle hypothesis is correct.
(B) Wallis's postulate is false.
(C) Euclidean geometry is consistent.
(D) Euclidean geometry is inconsistent.

D
(c) The property of Beltrami's pseudosphere model that makes it unique is
(A) It represents angles accurately.
(B) It represents the entire hyperbolic plane, not just a portion of it.
(C) Its lines are actually straight lines in the Euclidean sense.
(D) It can be realized by a 3-dimensional object in physical Euclidean space.

D
(d) The uniformity theorem shows that
(A) all Saccheri quadrilaterals have the same type of summit angles.
(B) the straight lines in the Klein model are chords of a circle.
(C) the perpendicular segments from one side of an acute angle to the other are unbounded.
(D) two lines are parallel if they are equidistant.

A
(e) Dehn's models show that
(A) elliptic parallelism is consistent with the Hilbert IBC axioms.
(B) if the sum of the angles in a triangle is always $180^{\circ}$, then Euclid's parallel postulate holds.
(C) hyperbolic parallelism is not exactly equivalent to the acute angle hypothesis.
(D) geometries that do not satisfy Dedekind's axiom are inconsistent.

C
2. (30 pts.) Rearrange these names into historical order, earliest to latest:

Euclid, Lambert, Archimedes, Poincaré, Bolyai (the son), Thales
Thales, Euclid, Archimedes, Lambert, Bolyai, Poincaré
3. (15 pts.) What is meant by a conformal model of hyperbolic geometry? Which of the 3 standard planar models is/are conformal?
In a conformal model, congruent angles in the hyperbolic space are represented by congruent angles in the Euclidean plane. The Poincaré disk and half-plane models are conformal; the Klein disk is not. (The hyperboloidal and Beltrami models are not planar.)
4. (15 pts.) Define the defect of a triangle; explain how it differs among Euclidean, hyperbolic, and elliptic geometries.
5. (20 pts.) State and prove ONE of these. If you try both, clearly indicate which one you want graded.
(A) Pasch's theorem.
(B) Under a suitable hypothesis (which you need to state), the sum of the angles in a triangle is $180^{\circ}$.
6. (Essay - 25 pts.)

State the Alternate Interior Angle Theorem and its converse. Tell which of them can be proved from the Hilbert IBC axioms and which cannot. Explain which of them excludes elliptic geometry and which excludes hyperbolic geometry. Feel free to add relevant comments.
7. (20 pts.) Do ONE of these. If you try both, clearly indicate which one you want graded. (Continue on back if necessary.)
(A) In the Poincaré disk model, the formula for arc length is

$$
d s^{2}=\frac{4\left(d x^{2}+d y^{2}\right)}{\left[1-\left(x^{2}+y^{2}\right)\right]^{2}}
$$

Introduce polar coordinates $(\rho, \theta)$ in the $(x, y)$ plane and show that the further coordinate transformation

$$
\rho=\tanh \left(\frac{r}{2}\right)
$$

converts the arc length to

$$
d s^{2}=d r^{2}+\sinh ^{2} r d \theta^{2}
$$

(B) Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The cross-ratio is defined as

$$
(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}} \frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}},
$$

where, for instance, $\overline{\mathrm{AB}}$ is the Euclidean distance between A and B . Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is additive in the sense that

$$
d(\mathrm{AB})=d(\mathrm{AC})+d(\mathrm{CB})
$$

when $\mathrm{A} * \mathrm{C} * \mathrm{~B}$ along a Poincaré line. (Verify that it is indeed additive.)

