Math. 467 (Fulling) 27 February 2019

Midterm Test – Solutions

1. *(Multiple choice – each 5 pts.) (Circle the correct capital letter.)*

   (a) The simplest example of an *elliptic* incidence geometry has

   (A) 3 points.
   (B) 4 points.
   (C) 5 points.
   (D) 7 points.
   (E) [none of these]

A

   (b) Which of the following is *not* provable from the Hilbert I and B axioms?

   (A) If D is in the interior of $\angle$CAB, then so is every other point on $\overrightarrow{\text{AD}}$ except A.
   (B) Every point D in the interior of $\angle$CAB lies on a segment joining a point E on $\overrightarrow{\text{AB}}$ to a point F on $\overrightarrow{\text{AC}}$.
   (C) If D lies on line $\overleftrightarrow{\text{BC}}$, then D is in the interior of $\angle$CAB if B $*$ D $*$ C.
   (D) If $\overrightarrow{\text{AD}}$ is between $\overrightarrow{\text{AC}}$ and $\overrightarrow{\text{AB}}$, then $\overrightarrow{\text{AD}}$ intersects segment BC.
   (E) If D lies on line $\overleftrightarrow{\text{BC}}$, then B $*$ D $*$ C if D is in the interior of $\angle$CAB.

B [the Warning, p. 115]

   (c) Which of these is indisputably true?

   (A) Babylonians were aware of the Pythagorean theorem 1000 years before Pythagoras.
   (B) The ancient Hebrews believed that π is exactly equal to 3.
   (C) Euclid's hostility to Aristotle's theory of the "finite universe" led him to his postulate that lines are actually infinitely long.
   (D) Euclid stated the parallel postulate in the form we most often use today.
   (E) Euclid's proof of the side-angle-side theorem is still valid in Hilbert's geometry.

A [See Greenberg, p. 2. (In C, "hostility" is sheer speculation.)]

   (d) Axiom C-4 authorizes us to construct an angle $\angle$B′A′C′ congruent to a given angle $\angle$BAC adjacent to a given ray $\overrightarrow{\text{A}'\text{B}'}$. The result is unique provided that

   (A) the *side* of the line $\overleftrightarrow{\text{A}'\text{B}'}$ on which the new angle appears is prescribed.
   (B) segment AB is congruent to segment A′B′.
   (C) the Euclidean parallel postulate holds.
   (D) $\angle$BAC is a right angle.
   (E) congruence of angles is an equivalence relation.

A

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2. *(10 pts.)* Simplify $\neg\forall x\,\exists y\;\big[\big(x>0\wedge x\le y\big)\vee\forall n\,\big(x>y^n\;\Rightarrow\;x+y>(x+y)^n\big)\big]$.
(Push the "$\neg$" in as far as you can!) (In Greenberg's notation, $\neg$ is $\sim$, and $\wedge$ is &.)

$$\exists x\,\forall y\,\neg\,\big[\big(x>0\wedge x\le y\big)\vee\forall n\,\big(x>y^n\;\Rightarrow\;x+y>(x+y)^n\big)\big]$$

$$\exists x\,\forall y\;\big[\neg\big(x>0\wedge x\le y\big)\wedge\exists n\,\neg\big(x>y^n\;\Rightarrow\;x+y>(x+y)^n\big)\big]$$

$$\exists x\,\forall y\;\big[\big(x\le 0\vee x>y\big)\wedge\exists n\,\big(x>y^n\wedge x+y\le(x+y)^n\big)\big]$$

3. *(15 pts.)* State the three Hilbert **incidence** axioms.

[See Greenberg, p. 69.]

4. *(20 pts.)* Do **ONE** of these [(A) or (B)]. *(Extra credit for doing both is limited to 10 points.)*

(A) State and prove Pasch's theorem.

[See Greenberg, p. 114.]

(B) Recall that Axiom B-4 states:

For any line l and any three points A, B, C not lying on l,

(i) If A and B are on the same side of l and if B and C are on the same side of l, then A and C are on the same side of l.

(ii) If A and B are on opposite sides of l and if B and C are on opposite sides of l, then A and C are on the same side of l.

Here is the question [(B)]:

(a) State the definitions of "opposite side" and "same side" that are in effect in Axiom B-4.

(b) Explain how Axiom B-4 (unlike the definitions you just stated) defines a "side" and shows that a line has exactly two sides.

[See Greenberg, pp. 110–112, and notes, pp. 23–24. You could use the language of equivalence relations and partitions, or not.]

5. *(15 pts.)* State the *crossbar theorem* and draw a sketch to illustrate it.

[See Greenberg, p. 116. The relevant sketch is the one at the *bottom* of the page.]

6. *(Essay – 20 pts.) IMPROVEMENTS IN THE REWRITE GET ONLY HALF CREDIT. E.G., IF YOUR INITIAL SCORE IS 16, YOUR FINAL SCORE WON'T EXCEED 18.*

Let $\mathcal{S}$ be a set. Recall that an *equivalence relation* on $\mathcal{S}$ is a binary relation $\sim$ with the properties (for all A, B, C in $\mathcal{S}$)
(1) reflexivity: $A \sim A$,
(2) symmetry: $A \sim B \ \Rightarrow\ B \sim A$,
(3) transitivity: $A \sim B \wedge B \sim C \ \Rightarrow\ A \sim C$.
Recall also that a *partition* of $\mathcal{S}$ is a collection $\{S_i\}$ of subsets of S with the properties
(4) disjointness: $S_i \cap S_j = \emptyset$ if $i \neq j$,
(5) exhaustiveness: $S = \bigcup_i S_i$.

Show that, given a partition of $\mathcal{S}$, the definition

$$A \sim B \iff A \text{ and } B \text{ belong to the same } S_i$$

defines an equivalence relation on $\mathcal{S}$. Then show that, conversely, every equivalence relation on $\mathcal{S}$ determines a partition of $\mathcal{S}$.