Math. 489GR (Fulling) 7 October 2009

Test A – Solutions

1. *(25 pts.)* If $\tilde{\omega}$ is a covector, then (by definition) $\tilde{\omega}(\vec{v})$ is a linear function of the vector argument $\vec{v}$ whose calculated values must be the same in all reference frames. Under a certain change of frame the components of vectors transform according to

$$v^{\beta'} = \Lambda^{\beta'}{}_{\alpha} v^{\alpha}.$$

Deduce the transformation law of the covector components, ω_α .

$$\omega_\alpha v^\alpha = \tilde{\omega}(\vec{v}) = \omega_{\beta'} v^{\beta'} = \omega_{\beta'} \Lambda^{\beta'}{}_{\alpha} v^{\alpha} \,,$$

so

$$\omega_\alpha = \omega_{\beta'} \Lambda^{\beta'}{}_{\alpha} \,.$$

Alternative method (credit John Langford): $v^{\beta'} = \Lambda^{\beta'}{}_{\alpha} v^{\alpha}$ is equivalent to the basis transformation $\vec{e}_\alpha = \Lambda^{\beta'}{}_{\alpha} \vec{e}_{\beta'}$. The components of $\tilde{\omega}$ are its values on the basis vectors:

$$\omega_\alpha = \tilde{\omega}(\vec{e}_\alpha) = \Lambda^{\beta'}{}_{\alpha} \tilde{\omega}(\vec{e}_{\beta'})$$

(by linearity), so

$$\omega_\alpha = \Lambda^{\beta'}{}_{\alpha} \omega_{\beta'} \,.$$

Remark: The formula can also be written either

$$\omega_{\beta'} = \left(\Lambda^{-1}\right)^{\alpha}{}_{\beta'} \omega_\alpha \qquad \text{or} \qquad \omega_{\beta'} = \Lambda^{\alpha}{}_{\beta'} \omega_\alpha \,.$$

In the last case we are using Schutz's convention that the direction of the transformation is indicated unambiguously by the positions of the primed and unprimed indices.

2. *(essay – 20 pts.)* Explain what *tides* have to do with general relativity. (If you were standing on a planet of very small radius, how would you see nearby objects fall, relative to each other and you?) Explain why a *uniform* gravitational field produces no tides.

[Cf. Exercise 5.2.]

3. *(20 pts.)* The 4-velocity of a rocket ship is $\vec{U} = (2, 1, 1, 1)$ (in earth's inertial frame). It encounters a cosmic ray (high-energy particle) whose 4-momentum is $\vec{P} = (30, 25, 0, 0) \times 10^{-26}$ kg .

   (a) What is the rest mass of the particle?

$\vec{P} \cdot \vec{P} = -m^2$, so

$$m = \sqrt{E^2 - \mathbf{p}^2} = \sqrt{900 - 625} = \sqrt{225} = 11\sqrt{5}$$

(in units 10^{-26} kg).

(b) What is the energy of the particle in the rocket's frame?

[Cf. Exercise 2.30.] Take inner product with the basis vector in the time direction in the rocket's frame:

$$E' = -\vec{U} \cdot \vec{P} = 60 - 25 = 35$$

(in units 10^{-26} kg).

4. *(35 pts.)* We define hyperbolic coordinates, (τ, σ), in two-dimensional space-time by

$$t = \sigma \sinh \tau, \qquad x = \sigma \cosh \tau.$$

Let T be a $\binom{2}{0}$ tensor field whose components in the hyperbolic frame are

$$T \xrightarrow{\mathcal{O}'} \begin{pmatrix} 2 & 0 \\ 0 & \tau \end{pmatrix}; \qquad \text{i.e.,} \quad T^{\tau\tau} = 2, \quad T^{\sigma\sigma} = \tau, \quad T^{\tau\sigma} = 0 = T^{\sigma\tau}.$$

(a) Find the components of T with respect to the inertial frame, (t, x). (Leave the answer as functions of the variables τ and σ.)

$$T^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu'} \Lambda^{\beta}{}_{\nu'} T^{\mu'\nu'}$$

where

$$\Lambda^{\alpha}{}_{\mu'} = \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial \sigma} \\ \frac{\partial x}{\partial \tau} & \frac{dx}{d\sigma} \end{pmatrix} = \begin{pmatrix} \sigma \cosh t & \sinh \tau \\ \sigma \sinh \tau & \cosh \tau \end{pmatrix}.$$

In matrix notation,

$$T' = \Lambda T \Lambda^{\mathrm{t}}$$

(since the ν' is connected to the *column* index of the second Λ). After two matrix multiplications,

$$T^{\alpha\beta} = \begin{pmatrix} 2\sigma^2 \cosh^2 \tau + \tau \sinh^2 \tau & (2\sigma^2 + \tau) \sinh \tau \cosh \tau \\ (2\sigma^2 + \tau) \sinh \tau \cosh \tau & 2\sigma^2 \sinh^2 \tau + \tau \cosh^2 \tau \end{pmatrix}.$$

Remark: In a real physical problem the matrix elements of T (and the arguments of the hyperbolic functions) would involve some physical constants to keep the units right. (A pressure is not a time, for example.)

(b) Explain what a $\binom{2}{0}$ tensor "really is" (the modern definition as a *multilinear functional* on some space).

[Talk about *bi*linear functionals on *covectors* ($T\colon \mathcal{V}^* \times \mathcal{V}^* \to \mathbf{R}$, linear in each argument with the other argument fixed).]