Solutions to Exercises 2.19 and 2.21

2.19(a), *Method 1a* (credit John Cesar and others): We are given that $\vec{a}$ has constant spatial direction, so its spatial components at any time are the same as at any other time except possibly for an overall multiplicative constant:

$$\vec{a}(t_1) = (a^0, a^1, a^2, a^3), \quad \vec{a}(t_2) = (b^0, Ca^1, Ca^2, Ca^3).$$

According to (2.32), $\vec{U} \cdot \vec{a} = 0$; it follows that in the body's MCRF, $a^0 = 0 = b^0$. But we also are given that $\vec{a}$ always has the same magnitude, so we get

$${a_1}^2 + {a_2}^2 + {a_3}^2 = C^2({a_1}^2 + {a_2}^2 + {a_3}^2).$$

Thus $|C| = 1$, and to preserve direction $C = +1$. Thus at all times

$$\vec{a}(t) = (0, a^1, a^2, a^3) \quad \text{in MCRF},$$

and its spatial part is the Galilean acceleration.

Method 1b (credit Sean Grant and others):

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma \left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

$$\begin{aligned}\vec{a} = \frac{d\vec{U}}{d\tau} &= \gamma \left[\frac{d\gamma}{dt}\left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) + \gamma\left(0, \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2}\right)\right] \\ &= \gamma \left[\frac{d\gamma}{dt}\begin{pmatrix}1\\ \mathbf{v}\end{pmatrix} + \gamma\begin{pmatrix}0\\ \mathbf{a}\end{pmatrix}\right],\end{aligned}$$

where $\mathbf{a}$ is the Galilean acceleration. Furthermore,

$$\gamma = (1 - v^2)^{-1/2} \;\Rightarrow\; \frac{d\gamma}{dt} = v\,\frac{dv}{dt}\,(1 - v^2)^{-3/2}.$$

If we are in the MCRF, $\gamma = 1$ and $\mathbf{v} = 0$ and hence $\frac{d\gamma}{dt} = 0$. So we end up with

$$\vec{a} = \begin{pmatrix}0\\ \mathbf{a}\end{pmatrix} \quad \text{in MCRF.}$$

Method 2: Do ***2.21*** *first.*

$$\frac{d}{d\lambda}\begin{pmatrix}t\\ x\end{pmatrix} = \begin{pmatrix}\cosh(\lambda/a)\\ \sinh(\lambda/a)\end{pmatrix} \equiv \vec{U}.$$

Note that $\vec{U} \cdot \vec{U} = -1$, so we can identify $\vec{U}$ as the 4-velocity and λ as the proper time. Then

$$\frac{d}{d\tau}\vec{U} = \frac{d^2}{d\lambda^2}\begin{pmatrix}t\\ x\end{pmatrix} = \frac{1}{a}\begin{pmatrix}\sinh(\tau/a)\\ \cosh(\tau/a)\end{pmatrix},$$

which has constant magnitude a^{-2} (and constant spatial direction, since we're considering only one spatial dimension). Thus this worldline has uniform acceleration, $\alpha = 1/a$, The Lorentz transformation into the MCRF (mapping $\vec{U}$ to $(1,0)$) is

$$\Lambda = \begin{pmatrix} \cosh(\lambda/a) & -\sinh(\lambda/a) \\ -\sinh(\lambda/a) & \cosh(\lambda/a) \end{pmatrix}. \tag{1}$$

Applied to $\vec{a}$, Λ yields

$$\frac{1}{a}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix},$$

which again proves 2.19(a). (Clearly this worldline can be embedded into 4-dimensional space-time, the two extra dimensions remaining inert. Any other uniform-acceleration worldline can be put into this form by rotation and translation of coordinates.)

2.19(b), *Method 1: Use Ex. 2.21.* Without loss of generality we can write

$$t = \frac{1}{\alpha}\sinh(\alpha\tau), \qquad x = \frac{1}{\alpha}\cosh(\alpha\tau),$$

$$\vec{U} = \begin{pmatrix} \cosh(\alpha\tau) \\ \sinh(\alpha\tau) \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} \Rightarrow v = \frac{\sinh(\alpha\tau)}{\cosh(\alpha\tau)} = \tanh(\alpha\tau).$$

Solving, we have

$$\tau = \frac{1}{\alpha}\sinh^{-1}(\alpha t)$$

and hence

$$v = \tanh\bigl(\sinh^{-1}(\alpha t)\bigr). \tag{2}$$

We can get rid of the hyperbolic functions:

$$\begin{aligned} v &= \frac{\sinh(\sinh^{-1}(\alpha t))}{\cosh(\sinh^{-1}(\alpha t))} \\ &= \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}}. \end{aligned} \tag{3}$$

The inverse equation inverse to (3) (needed to answer the numerical question) is

$$t = \frac{1}{\alpha}\frac{v}{\sqrt{1 - v^2}}.$$

The change in x is

$$\Delta x = \frac{1}{\alpha}\cosh(\alpha\tau) - \frac{1}{\alpha}$$

(note that $x(0) = 1/\alpha$, not 0), so

$$\Delta x = \frac{1}{\alpha}[\cosh(\sinh^{-1}(\alpha t)) - 1] = \frac{1}{\alpha}[\sqrt{1 + (\alpha t)^2} - 1].$$

For $\alpha = 10 \text{ m/s}^2 = \frac{1}{3} \times 10^{-7} \text{ s}^{-1}$ and $v = 0.999$, one gets $t \approx \frac{2}{3} \times 10^9 \text{ s} \approx 20$ years. (A convenient fact to remember is that 1 year $\approx \pi \times 10^7$ seconds.) (Numerical answers on student papers were all over the lot, even among those who had the correct formulas.)

Method 2(a) (credit Robert DeAlba and others): (I suppress the two transverse dimensions.) Transform

$$\vec{U} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{a} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

from the MCRF back to the inertial frame by [in notation set by (1)]

$$\Lambda^{-1} = \begin{pmatrix} \cosh(\lambda/a) & \sinh(\lambda/a) \\ \sinh(\lambda/a) & \cosh(\lambda/a) \end{pmatrix},$$

getting

$$\vec{U} = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}, \qquad \begin{pmatrix} v\gamma\alpha \\ \gamma\alpha \end{pmatrix} = \vec{a} = \frac{d\vec{U}}{d\tau}\,.$$

Thus

$$\frac{d\gamma}{d\tau} = v\gamma\alpha, \qquad \frac{d}{d\tau}(\gamma v) = \gamma\alpha.$$

From either of these equations and the definition of γ you get after several steps of calculus and algebra

$$\alpha = \gamma^2 \frac{dv}{d\tau}\,.$$

Therefore,

$$\alpha \int d\tau = \int \gamma^2\, dv = \int \frac{dv}{1 - v^2}\,,$$

$$\alpha\tau = \tanh^{-1} v \;\Rightarrow\; v = \tanh(\alpha\tau). \tag{4}$$

Since

$$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} \equiv \frac{U^1}{U^0}$$

and $\vec{U}$ is normalized to -1, it follows that

$$\vec{U} = \begin{pmatrix} \cosh(\alpha\tau) \\ \sinh(\alpha\tau) \end{pmatrix}.$$

Hence

$$\frac{dt}{d\tau} = U^0 = \cosh(\alpha\tau) = \Rightarrow\; t = \frac{1}{\alpha}\sinh(\alpha\tau),$$

so (2) follows from (4) and you can proceed as in Method 1.

There were several variations on this theme that somehow led directly to (3) without going through (2).

2.19(c): In either approach above, we already know the relations among τ, t, and x.

From $\tau = \alpha^{-1}\sinh^{-1}(\alpha t)$ and the numerical values of α and t above, we get $\tau = 1.14\times10^{8}$ s $= 3.6$ years. Then from either formula for Δx above we get $\Delta x = 6.37\times10^{8}$ s $= 1.9\times10^{17}$ m.

For the center of the galaxy, we have $\Delta x = 6.7\times10^{11}$ s and

$$\tau = \frac{1}{\alpha}\cosh^{-1}(1+\alpha\Delta x) = 3.2\times10^{8}\ \text{s} \approx 10\ \text{years}.$$