Adjoints *(Sec. 18)*

Adjoints are defined only when there is an inner product!

The *adjoint* of $\underline{A}$ is a linear operator, $\underline{A}^*$, such that

$$\vec{u} \cdot (\underline{A}\vec{v}) = (\underline{A}^*\vec{u}) \cdot \vec{v} \qquad \text{— or } \langle \vec{u}, \underline{A}\vec{v} \rangle = \langle \underline{A}^*\vec{u}, \vec{v} \rangle.$$

More precisely:

Definition: Let $\mathcal{V}$ and $\mathcal{U}$ be inner product spaces (possibly the same!), and $\underline{A} : \mathcal{V} \to \mathcal{U}$ be linear. Consider a (temporarily) fixed $\vec{u} \in \mathcal{U}$. Suppose there exists a $\vec{w} \in \mathcal{V}$ such that

$$(\underline{A}\vec{v}) \cdot \vec{u} = \vec{v} \cdot \vec{w}, \qquad \forall \vec{v} \in \mathcal{V}.$$

Then $\vec{w}$ is denoted by $\underline{A}^*\vec{u}$. [$\vec{w}$ is unique if it exists, since $\vec{v} \cdot \vec{w}_1 = \vec{v} \cdot \vec{w}_2 \quad \forall \vec{v} \in \mathcal{V} \;\Rightarrow\; \vec{v} \cdot (\vec{w}_1 - \vec{w}_2) = 0 \quad \forall \vec{v} \;\Rightarrow\; \vec{w}_1 = \vec{w}_2$ by Ex. 12.5.] This construction defines an operator $\underline{A}^*$ whose domain is the subset of $\vec{u} \in \mathcal{U}$ for which such $\vec{w}$'s exist, and whose codomain is $\mathcal{V}$.

$$\mathcal{V} \;\underset{\underline{A}^*}{\overset{\underline{A}}{\rightleftarrows}}\; \mathcal{U}$$

Remark: When we get to linear functionals (Chap. 7), we'll see that dom $\underline{A}^* =$ all of $\mathcal{U}$ if

i) $\mathcal{V}$ and $\mathcal{U}$ are finite-dimensional, or

ii) $\mathcal{V}$ and $\mathcal{U}$ are infinite-dimensional Hilbert spaces, and $\underline{A}$ is a bounded (= continuous) operator.

Theorem. *dom $\underline{A}^*$ is a sub***space** *of $\mathcal{U}$, and $\underline{A}^*$ is linear.*

Proof: Suppose $\vec{w}_1 = \underline{A}^*\vec{u}_1, \quad \vec{w}_2 = \underline{A}^*\vec{u}_2$. Then

$$\vec{v} \cdot (\lambda\vec{w}_1 + \vec{w}_2) = \overline{\lambda}\,\vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2 = \overline{\lambda}(\underline{A}\vec{v}) \cdot \vec{u}_1 + (\underline{A}\vec{v}) \cdot \vec{u}_2 = (\underline{A}\vec{v}) \cdot (\lambda\vec{u}_1 + \vec{u}_2).$$

Thus $\underline{A}^*(\lambda\vec{u}_1 + \vec{u}_2)$ exists and equals $\lambda\vec{w}_1 + \vec{w}_2$, QED.

Theorem 18.1$'$. *Let $\mathcal{V}$ and $\mathcal{U}$ be finite-dimensional. Then, with respect to* **orthonormal** *bases, the matrix of $\underline{A}^*$ is [the complex conjugate of] the transpose of the matrix of $\underline{A}$. More precisely, let $\{A^j{}_k\} \equiv \{A_{jk}\}$ be the matrix representing $\underline{A}$ with respect to given ON bases for $\mathcal{V}$ and $\mathcal{U}$. Then the matrix of $\underline{A}^*$ is $\{(A^*)^j{}_k\} = \{\overline{A_{kj}}\}$.*

REMARK: In $\{A^j{}_k\}$, k runs from 1 to $\dim \mathcal{V} \equiv n$,
j runs from 1 to $\dim \mathcal{U} \equiv m$.
In $\{(A^*)^j{}_k\}$, k runs from 1 to m,
j runs from 1 to n.

Another convention, which is more lucid in certain respects, is to associate the same index letter always with the same space (say j with $\mathcal{U}$ and k with $\mathcal{V}$). (Bowen & Wang do that.) Here, I have chosen the convention which points up the transposition property most sharply.

PROOF: Since the bases are fixed, the theorem reduces to a statement about linear maps between $\mathcal{F}^n$ and $\mathcal{F}^m$. With this identification,

$$(\underline{A}\vec{v})^j = A^j{}_k\, v^k, \qquad \text{[summation implied]}$$

$$(\underline{A}\vec{v}) \cdot \vec{u} = \overline{u^j}\, A^j{}_k\, v^k = \overline{(\overline{A^j{}_k}\, u^j)}\, v^k = \vec{v} \cdot \vec{w},$$

where $w^k = \overline{A^j{}_k}\, u^j$. Since $\vec{w} \equiv \underline{A}^*\vec{u}$, this says that $(A^*)^k{}_j = \overline{A^j{}_k}$. Interchanging the meaning of j and k (cf. Remark!), we obtain the formula stated in the theorem.

THEOREM 18.2. *Among other properties* (see book),

(b) $(\underline{A}\,\underline{B})^* = \underline{B}^*\underline{A}^*$, *(c)* $(\lambda\underline{A})^* = \overline{\lambda}\underline{A}^*$,
(f) $(\underline{A}^*)^* = \underline{A}$, *(g)* $(\underline{A}^*)^{-1} = (\underline{A}^{-1})^*$.

(In each case we make the obvious technical assumptions, such as that $\operatorname{dom} \underline{A}^* = \operatorname{codom} \underline{A}$, *that* $\underline{A}^{-1}$ *exists in (g), etc. For unbounded operators in infinite-dimensional Hilbert spaces, the domain of* $(\underline{A}^*)^*)$ *can be larger than the domain of* $\underline{A}$, *so (f) is false in that case.)*

PROOF OF (b): $\underline{B} : \mathcal{V} \to \mathcal{U}$, $\underline{A} : \mathcal{U} \to \mathcal{W}$. Therefore $\underline{B}^* : \mathcal{U} \to \mathcal{V}$, $\underline{A}^* : \mathcal{W} \to \mathcal{U}$, so $\underline{B}^*\underline{A}^*$ makes sense ($\mathcal{W} \to \mathcal{V}$) — whereas $\underline{A}^*\underline{B}^*$ generally doesn't.

$$\vec{v} \cdot (\underline{B}^*\underline{A}^*\vec{w}) = (\underline{B}\vec{v}) \cdot (\underline{A}^*\vec{w}) = (\underline{A}\underline{B}\vec{v}) \cdot \vec{w} \qquad (\forall \vec{v} \in \mathcal{V},\ \forall \vec{w} \in \mathcal{W}).$$

Thus $\underline{B}^*\underline{A}^*\vec{w} = (\underline{A}\,\underline{B})^*\vec{w}$, QED.

PROOF OF (g): $\underline{A} : \mathcal{V} \to \mathcal{U}$ is assumed invertible. Note:

$$\underline{A}^* : \mathcal{U} \to \mathcal{V}, \qquad \underline{A}^{-1} : \mathcal{U} \to \mathcal{V}, \qquad \underline{A}^{*-1} : \mathcal{V} \to \mathcal{U}.$$

Let $\underline{B} = \underline{A}^{-1}$ in (b): $(\underline{A}^{-1})^*\underline{A}^* = (\underline{A}\,\underline{A}^{-1})^* = \underline{1}^* = \underline{1}$, and similarly $\underline{A}^*(\underline{A}^{-1})^* = \underline{1}$. Therefore $(\underline{A}^*)^{-1}$ exists and equals $(\underline{A}^{-1})^*$.

Theorem 18.3′. $\underline{A} : \mathcal{V} \to \mathcal{U}$ *linear* $\Rightarrow$

(1) $\ker \underline{A}^* = (\operatorname{ran} \underline{A})^\perp \quad (\subset \mathcal{U})$

(2) $\ker \underline{A} = (\operatorname{ran} \underline{A}^*)^\perp \quad (\subset \mathcal{V})$ (in finite dimensions, or under other conditions that guarantee $\underline{A}^{**} = \underline{A}$.)

PROOF:

(1) $\vec{u} \in (\operatorname{ran} \underline{A})^\perp \iff \forall \vec{v} \in \mathcal{V},\ 0 = \vec{u} \cdot (\underline{A}\vec{v}) = (\underline{A}^*\vec{u}) \cdot \vec{v} \iff \underline{A}^*\vec{u} = \vec{0}$ (Ex. 12.5) $\iff \vec{u} \in \ker \underline{A}^*$.

(2) Let $\underline{A}$ in (1) be $\underline{A}^*$; use $\underline{A}^{**} = \underline{A}$.

Corollary 1 (Theorem 18.3). *If* ran $\underline{A}$ *is a closed set (always true if* ran $\underline{A}$ *is finite-dimensional), then*

$$\operatorname{ran} \underline{A} = (\ker \underline{A}^*)^\perp$$

and

$$\mathcal{U} = \ker \underline{A}^* \oplus \operatorname{ran} \underline{A}.$$

(Cf. Theorem 13.4′ and following remarks in notes.) *Similarly,* $\mathcal{V} = \ker \underline{A} \oplus \operatorname{ran} \underline{A}^*$.

Corollary 2. *If* ran $\underline{A}$ *is closed,* $\underline{A}\vec{v} = \vec{b}$ *has solutions iff* $\vec{b}$ *is orthogonal to every solution of* $\underline{A}^*\vec{u} = \vec{0}$.

Recall that in finite-dimensional calculations with orthogonal bases, $\underline{A}^*$ will be represented by the complex-conjugate transpose of the matrix A.

Theorem 18.4. $\mathcal{V}$ *finite-dimensional* $\Rightarrow$ $\underline{A}$ *and* $\underline{A}^*$ *have the same rank. (In matrix terms: The column rank of* $\underline{A}$ *equals the row rank of* $\underline{A}$*.)*

PROOF: Corollary 1 $\Rightarrow$ $\dim \mathcal{V} = \dim\ker \underline{A} + \dim\operatorname{ran} \underline{A}^*$. (†)
Theorem 15.8 $\Rightarrow$ $\dim \mathcal{V} = \dim\ker \underline{A} + \dim\operatorname{ran} \underline{A}$. Thus $\dim \operatorname{ran} \underline{A}^* = \dim\operatorname{ran} \underline{A}$.

COROLLARY 3 **(Finite-dimensional Index Theorem).**

$$\begin{aligned}\dim\ker \underline{A} - \dim\ker \underline{A}^* &\equiv \operatorname{index} \underline{A} \\ &= \dim \mathcal{V} - \dim \mathcal{U}, \qquad \forall \underline{A} \in \mathcal{L}(\mathcal{V};\mathcal{U}).\end{aligned}$$

PROOF: Theorem 15.8 $\Rightarrow$ $\dim \mathcal{U} = \dim\ker \underline{A}^* + \dim\operatorname{ran} \underline{A}^*$. Subtract this equation from (†) above.

Corollary 4 (Finite-dimensional Fredholm Alternative Theorem). *Let* $\dim \mathcal{V} = \dim \mathcal{U} < \infty$. *Then* **either**

(A) $\underline{A}\vec{v} = \vec{b}$ *is solvable, uniquely, for all* $\vec{b}$, **or**

(B) $\underline{A}\vec{v} = \vec{0}$ *has* k *linearly independent nontrivial solutions* $(k \neq 0)$;
$\underline{A}^*\vec{u} = \vec{0}$ *also has* k *nontrivial solutions* $\{\vec{u}_1, \ldots, \vec{u}_k\}$*; and*
$\underline{A}\vec{v} = \vec{b}$ *has solutions (nonunique) iff* $\vec{b}$ *is orthogonal to* $\{\vec{u}_1, \ldots, \vec{u}_k\}$.

Special case: $\mathcal{V} = \mathcal{U}$, $\quad \underline{A}^* = \underline{A}$. *Then* $\underline{A}\vec{v} = \vec{b}$ *is solvable iff* $\vec{b}$ *is orthogonal to all the solutions of* $\underline{A}\vec{v} = \vec{0}$.

Corollary 1 leads to similar "alternative theorems" for certain classes of operators on infinite-dimensional Hilbert spaces. The next few lectures will discuss, nonrigorously, some examples.

Example 1: Fredholm integral operators

Reference: I. Stakgold, *Green's Functions and Boundary Value Problems* (Wiley, 1979), Chaps. 5 and 6.

Let $K(x,y)$ be a continuous function on the square $a \le x, y \le b \quad (|a|, b < \infty)$. [This condition is stronger than necessary.] Consider the equation

$$f(x) - \int_a^b K(x,y)\, f(y)\, dy = g(x), \tag{$*$}$$

(g given, f to be found.)

Fredholm Alternative Theorem. Either

(A) $(*)$ *has exactly one solution for each* $g \in \mathcal{L}^2(a,b)$. *(In particular,* $g = 0 \ \Rightarrow\ f = 0$.*)*

or

(B) $(*)$ *with* $g = 0$ *has* k *linearly independent solutions* $(k < \infty)$*; then the equation*

$$u(y) - \int_a^b \overline{K(x,y)}\, u(x)\, dx = 0 \tag{$*^*$}$$

also has k *solutions, and* $(*)$ *has solutions iff* g *is orthogonal to all of the latter.*

In more abstract notation: Let $\underline{K}$ *be the operator defined by*

$$[\underline{K}f](x) \equiv \int_a^b K(x,y)\, f(y)\, dy.$$

Then the adjoint $\underline{K}^*$ is given by

$$[\underline{K}^* u](y) = \int_a^b \overline{K(x,y)}\, u(x)\, dx$$

(i.e., the function $K(x,y)$ acts exactly like a matrix). The theorem says that

$$\dim\ker(\underline{1} - \underline{K}) = \dim\ker(\underline{1} - \underline{K}^*) < \infty,$$

and that $(\underline{1} - \underline{K})f = g$ is solvable [i.e., $g \in \operatorname{ran}(\underline{1} - \underline{K})$] if and only if

$$g \cdot h \equiv \int_a^b g(x)\, \overline{h(x)}\, dx = 0, \quad \forall h \in \ker(\underline{1} - \underline{K}^*).$$

(This last part is equivalent to saying that $\operatorname{ran}(\underline{1} - \underline{K})$ is **closed***; cf. Corollary 1.)*

PROOF omitted.

A practical consequence of this theorem is that if a uniqueness theorem can be proved for the integral equation in question, then one knows that alternative (A) applies, and therefore a solution exists for all g. Since uniqueness proofs are usually easier than existence proofs, this property can be quite valuable.

The theorem can be extended to the case where x and y vary over a bounded domain in $\mathbf{R}^n$ (rather than a bounded interval in $\mathbf{R}$). Certain *differential* equations plus boundary conditions can be converted into integral equations of this type, as we'll see next.

FREDHOLM OPERATORS: The terminology in the literature is somewhat ambiguous: An "operator of Fredholm type" is not the same thing as an operator for which the "Fredholm alternative" holds. A Fredholm operator in the latter, narrower, sense is one of the form $\underline{A} = \underline{1} + \underline{K}$, where $\underline{K}$ is *compact*. I will not be able to define "compact" in this course; suffice it to say that this is the property which the boundedness of the interval in the foregoing example was sufficient to ensure. For such an operator it can be proved that $\dim\ker\underline{A} = \dim\ker\underline{A}^* < \infty$ (i.e., index $\underline{A} = 0$), and also that $\operatorname{ran}\underline{A}$ is closed. Therefore, the Fredholm alternative holds; *"Uniqueness implies Existence."* These operators are analogous to finite-dimensional operators represented by *square* matrices, as described in Corollary 4. Note that the domain and codomain must be the same space (or at least isomorphic) so that the identity operator is defined.

An operator $\underline{A}$ is said to be of Fredholm type if there is an operator $\underline{B}$ such that $\underline{B}\,\underline{A} = \underline{1} + \underline{K}_1$ and $\underline{A}\,\underline{B} = \underline{1} + \underline{K}_2$ for some compact $\underline{K}_1$ and $\underline{K}_2$. For these operators it can be shown that $\operatorname{ran}\underline{A}$ is closed, and that $\dim\ker\underline{A} < \infty$, $\dim\ker\underline{A}^* < \infty$ (i.e., index $\underline{A}$ is defined and finite, but not necessarily zero). These operators are analogous to operators between finite-dimensional spaces whose dimensions are not necessarily the same — i.e., operators represented by rectangular matrices — as described in the propositions preceding Corollary 4. The domain and codomain of $\underline{A}$ don't have to be the same in this case.

Fredholm operators — a summary

Setting:	domain = codomain ($\underline{A}$ an endomorphism)	domain $\neq$ codomain ($\underline{A}$ a linear map)
Index:	zero	integer (maybe zero)
Finite dimensions:	square matrix	rectangular matrix
Infinite-dimensional Hilbert space:	$\underline{A} = \underline{1} + \underline{K}$, $\underline{K}$ compact	$\exists \underline{B}\colon \underline{B}\,\underline{A} = \underline{1} + \underline{K}_1$, $\underline{A}\,\underline{B} = \underline{1} + \underline{K}_2$, $\underline{K}_1$, $\underline{K}_2$ compact
Conclusions:	dim ker $\underline{A}$ = dim ker $\underline{A}^*$ dim ker $\underline{A} < \infty$ ran $\underline{A}$ closed	dim ker $\underline{A} < \infty$ dim ker $\underline{A}^* < \infty$ ran $\underline{A}$ closed
Terminology:	Fredholm alternative holds	Operator of Fredholm type

Example 2: An ordinary differential operator

Recall that in homework we saw that

$$\underline{A}f(t) \equiv f''(t) + \omega^2 f(t) = g(t), \qquad 0 < t < 1,$$

$$f(0) = 0 = f(1), \tag{C$_1$}$$

has solutions if and only if g is orthogonal to the solution (if any) of the corresponding homogeneous problem; that is, if $\omega = n\pi$, then

$$\int_0^1 g(t)\,\sin n\pi t\,dt = 0$$

is necessary (and sufficient) for solvability. (We assume $g \in \mathcal{L}^2(0,1)$ at worst.) This cries out to be interpreted as an example of the theorem that ran $\underline{A} = (\ker\,\underline{A}^*)^\perp$ (or, at least,

$$\operatorname{ran}\,\underline{A} \subseteq \overline{\operatorname{ran}\,\underline{A}} = (\operatorname{ran}\,\underline{A})^{\perp\perp} = (\ker\,\underline{A}^*)^\perp,$$

where the overlining indicates the topological closure of the set (i.e., its limit points are added); this weaker condition, which was all that Theorem 18.3′ gave us in general, indicates the necessity but not the sufficiency of such an orthogonality condition).

A rigorous treatment of this example is too technical to get into here, largely because $\underline{A}$ is an *unbounded* (discontinuous) *operator*, which *can't* even *be defined on all of the*

Hilbert space $\mathcal{L}^2(0,1)$. Suffice it to say that the adjoint of such an operator (including the boundary condition (C_1) as part of the definition of the operator, since it restricts the domain) can be defined, and that in this case, if $\omega^2 \in \mathbf{R}$, one can show

$$\underline{A}^* = \underline{A}.$$

Thus ran $\underline{A} \subseteq (\ker\, \underline{A})^\perp$ (in fact, they are equal), as we observed by direct calculation in the exercise.

Remark: $\underline{A}^* = \underline{A}$ is related to, but stronger than, the fact that

$$h \cdot (\underline{A}f) = (\underline{A}h) \cdot f$$

for all h and f satisfying (C_1). This is proved by integration by parts, the BC being needed to make the endpoint terms vanish.

Alternative approach to this example: Suppose we didn't know anything about trigonometric functions, but did know how to solve the ODE in the case $\omega = 0$:

$$f''(t) = g(t), \qquad f(0) = 0 = f(1).$$

Answer: $f(t) = \displaystyle\int_0^1 K(t,s)\, g(s)\, ds,$

$$K(t,s) \equiv \begin{cases} s(t-1) & \text{for } s \le t, \\ t(s-1) & \text{for } s \ge t. \end{cases}$$

Check: $t = s \Rightarrow$ consistency. $f(0) = 0, \quad f(1) = 0.$

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left[\int_0^t s(t-1)\, g(s)\, ds + \int_t^1 t(s-1)\, g(s)\, ds\right] \\ &= \int_0^t s\, g(s)\, ds + \int_t^1 (s-1)\, g(s)\, ds. \end{aligned}$$

$$f''(t) = t\, g(t) - (t-1)\, g(t) = g(t).$$

Apply this to our original equation by replacing g by $g - \omega^2 f$:

$$f(t) = \int_0^1 K(t,s)\, g(s)\, ds - \omega^2 \int_0^1 K(t,s)\, f(s)\, ds.$$

This is not a solution, since f still appears in the right-hand side. It *is* a Fredholm integral equation for f:

$$f(t) + \int_0^1 \omega^2 K(t,s)\, f(s)\, ds = \int_0^1 K(t,s)\, g(s)\, ds$$

is of the form

$$(\underline{1} + \omega^2 \underline{K})\, f(t) = h(t).$$

So the Fredholm alternative theorem applies: Note that $\overline{K(t,s)} = K(s,t)$, so $\underline{K}^* = \underline{K}$. Therefore solutions f exist $\big(h \in \operatorname{ran}\,(\underline{1} + \omega^2 \underline{K})\big)$ iff $h \perp \ker\,(\underline{1} + {}^{(}\overline{\omega^2}{}^{)}\, \underline{K})$. The elements of this kernel are the solutions of the original homogeneous problem (nontrivial iff $\omega = n\pi$).

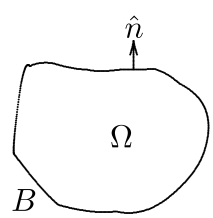
A more realistic application of this idea: Consider

$$f''(t) + \omega^2 f(t) - V(t) f(t) = g(t) \qquad \text{with BC } (\mathrm{C}_1),$$

where $\omega \neq n\pi$. Let $\underline{K}$ be the integral operator which solves $f'' + \omega^2 f = g$ with those same boundary conditions. (This integral kernel was constructed in homework). Thus $f(t) = \int_0^1 K(t,s) g(s)\, ds$. Replace g by $g + Vf$ to get a Fredholm integral equation equivalent to the differential equation (+ BC) we're interested in. Thus we can discuss existence and uniqueness questions for this boundary-value problem without writing down explicit solutions of $f'' + \omega^2 f - Vf = 0$ (which would be hard for general V).

Example 3: A partial differential operator

Instead of an inhomogeneous differential equation, we can have inhomogeneous boundary conditions.



Let $\Omega \subset \mathbf{R}^3$ be a bounded region with a smooth boundary, B. Consider the heat equation $\dfrac{\partial u}{\partial t} = \nabla^2 u$ in Ω with boundary condition

$$\frac{\partial u}{\partial n}(\vec{x}) = g(\vec{x}) \quad \text{if } \vec{x} \in B.$$

($\partial u / \partial n =$ normal derivative $\equiv \hat{n} \cdot \nabla u$.)

Physics $\Rightarrow$ $\frac{\partial u}{\partial n} \propto$ heat flow through B at $\vec{x}$. ($u =$ temperature.)

Let's look for *time-independent* solutions. (Math. 602 $\Rightarrow$ this is the complementary problem to the time-dependent problem with $\frac{\partial u}{\partial n} = 0$ on B and $u(\vec{x})$ given at $t = 0$.) We get Laplace's equation:

$$\nabla^2 u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } B \quad (g \text{ given}).$$

Observation: Gauss's theorem $\Rightarrow$

$$\begin{aligned}\int_B g\,dS \equiv \int_B \nabla u\cdot\hat{n}\,dS \equiv \int_B \nabla u\cdot d\vec{S}\\ = \int_\Omega \nabla\cdot(\nabla u)\,d^3x \equiv \int_\Omega \nabla^2 u\,d^3x = 0.\end{aligned}$$

Therefore, $\int_B g\,dS = 0$ is a necessary condition for existence of a solution!

Physical interpretation: $\int_B g\,dS = \int_B \frac{\partial u}{\partial n}\,dS$ is the net heat flow out of Ω. If this is not zero, naturally the temperature inside Ω can't be independent of time.

More generally, for the doubly inhomogeneous problem

$$\nabla^2 u = j(\vec{x}) \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g(\vec{x}) \text{ on } B$$

one has the consistency condition

$$\int_B g\,dS = \int_\Omega j\,d^3x. \tag{\#}$$

(Heat produced inside must balance heat flowing out.)

Fredholm-like interpretation: Consider the homogeneous problem

$$\nabla^2 u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } B.$$

One solution of this is $u =$ constant. (#) says the data (j, g) are orthogonal, in some sense, to the constant solutions of the homogeneous problem.

To see this, observe that the operator we are dealing with here really is

$$\underline{A} : \bigl(\text{some domain} \subset \mathcal{L}^2(\Omega)\bigr) \to \mathcal{L}^2(\Omega)\times\mathcal{L}^2(B),$$

$$\underline{A}u \equiv \left(\nabla^2 u, \frac{\partial u}{\partial n}\right).$$

We endow $\mathcal{L}^2(\Omega)\times\mathcal{L}^2(B) \cong \mathcal{L}^2(\Omega)\oplus\mathcal{L}^2(B)$ with the inner product which makes it an *orthogonal* direct sum: Let (f_1, h_1) be a generic element of the space. [Note: h_1 is independent of f_1, not its boundary value.] Then define

$$(f_1, h_1)\cdot(f_2.h_2) \equiv \int_\Omega f_1(\vec{x})\,\overline{f_2(\vec{x})}\,d^3x + \int_B h_1(\vec{x})\,\overline{h_2(\vec{x})}\,dS.$$

What is $\underline{A}^*$? $\int_\Omega [\underline{A}^*(f_1,h_1)](\vec{x})\,\overline{u(\vec{x})}\,d^3x$

$$\begin{aligned}
&\equiv [\underline{A}^*(f_1,h_1)]\cdot\vec{u} \quad = \quad (f_1.h_1)\cdot(\underline{A}\vec{u}) \\
&= \int_\Omega f_1(\vec{x})\,\overline{\nabla^2 u(\vec{x})}\,d^3x + \int_B h_1(\vec{x})\,\overline{\frac{\partial u}{\partial n}}\,dS \\
&= -\int_\Omega \nabla f_1\cdot\overline{\nabla u}\,d^3x + \int_B f_1(\vec{x})\,\overline{\nabla u(\vec{x})}\cdot d\vec{S} + \int_B h_1\,\overline{\nabla u}\cdot d\vec{S} \\
&= +\int_\Omega \nabla^2 f_1(\vec{x})\,\overline{u(\vec{x})}\,d^3x - \int_B \overline{u(\vec{x})}\,\nabla f_1(\vec{x})\cdot d\vec{S} + \int_B (f_1+h_1)\,\overline{\nabla u}\cdot d\vec{S}.
\end{aligned}$$

This will be consistent only if we take

$$\operatorname{dom}\underline{A}^* = \{\,(f_1,h_1) : \hat{n}\cdot\nabla f_1 = 0 \text{ on } B,\quad -f_1 = h_1 \text{ on } B,\quad \nabla^2 f_1 \text{ definable}\}$$

and take $\underline{A}^*(f_1,h_1) \equiv \nabla^2 f_1$. [In the last clause of the description of the domain, technicalities have been suppressed. The proper domain is slightly larger than $\mathcal{C}^2$, as far as smoothness in the interior of Ω is concerned.] Then

$$\ker\underline{A}^* = \{\,(f_1,h_1) : \hat{n}\cdot\nabla f_1 = 0 \text{ on } B,\quad -f_1 = h_1 \text{ on } B,\quad \nabla^2 f_1 = 0 \text{ in } \Omega\}.$$

This contains (in fact, consists entirely of) the constant functions (f_1,h_1) with $h_1 = -f_1$. Then, finally, $\operatorname{ran}\underline{A} \subset (\ker\underline{A}^*)^\perp \;\Rightarrow$

$$\int_\Omega j\,d^3x - \int_B g\,dS = 0 \qquad (\#)$$

for any $(j,g) \in \operatorname{ran}\underline{A}$. This is what we expected.