Adjoints and domain technicalities

Let $\mathcal{V}$ and $\mathcal{U}$ be infinite-dimensional Hilbert spaces. Then:

Case 1: Operator defined everywhere. Let $\underline{A}$ be a linear operator with domain $\mathcal{V}$ (*all* of $\mathcal{V}$) and codomain $\mathcal{U}$; that is, $\underline{A}: \mathcal{V} \to \mathcal{U}$, or $\underline{A} \in \mathcal{L}(\mathcal{V};\mathcal{U})$. Then, under an additional technical assumption, the adjoint $\underline{A}^*: \mathcal{U} \to \mathcal{V}$ is defined (with domain equal to *all* of $\mathcal{U}$). All the relations familiar for finite-dimensional adjoints then apply, such as $\underline{A}^{**} = \underline{A}$ and $(\underline{AB})^* = \underline{B}^* \underline{A}^*$.

The technical condition is that $\underline{A}$ be *bounded*. From our present point of view, the simplest way to define boundedness is simply to say that $\underline{A}$ is *continuous* as a mapping from $\mathcal{V}$ into $\mathcal{U}$, when the respective Hilbert-space norms are used to define distances (and hence convergence or ϵ-neighborhoods) in $\mathcal{V}$ and $\mathcal{U}$.

Most of the special classes of operators about which we need to prove theorems in this course, such as projections and isometries, are in fact always bounded and everywhere defined. Therefore, we can deal freely with their adjoints without worrying about domain technicalities.

Case 2: Operator not everywhere defined. Let $\underline{A}$ be a linear operator with codomain $\mathcal{U}$ which is defined only on some subspace (dom $\underline{A}$) of $\mathcal{V}$. (For instance, the second-derivative operator is defined on $\mathcal{C}^2(0,1) \subset \mathcal{L}^2(0,1)$.) Then, under an additional technical assumption, $\underline{A}^*$ is defined on some domain in $\mathcal{U}$.

The technical condition in this case is that dom $\underline{A}$ be *dense* in $\mathcal{V}$. Denseness can be defined by either of these equivalent conditions:

(i) $(\text{dom } A)^\perp = \{\vec{0}\}$

(ii) $\mathcal{V}$ is the closure of dom $\underline{A}$ (that is, every $\vec{v} \in \mathcal{V}$ is the limit of a sequence of vectors in the domain).

Even when dom $\underline{A}$ is dense, dom $\underline{A}^*$ may not be dense in $\mathcal{U}$, so $\underline{A}^{**}$ may not be definable. However, if $\underline{A}$ is Hermitian (which requires $\mathcal{U} = \mathcal{V}$), then dom $\underline{A}^*$ *includes* dom $\underline{A}$, so it is dense.