Tensors *(Chap. 7)*

Bilinear forms

Henceforth $\mathcal{F} = \mathbf{R}$. Consider a function of *two* variables in $\mathcal{V}$, which is linear in each variable with the other fixed:

$$\underline{Q}(\vec{u}, \vec{v}) \in \mathbf{R}, \qquad \underline{Q}\colon \mathcal{V} \times \mathcal{V} \to \mathbf{R},$$
$$\underline{Q}(\alpha \vec{u}_1 + \vec{u}_2, \vec{v}) = \alpha \underline{Q}(\vec{u}_1, \vec{v}) + \underline{Q}(\vec{u}_2, \vec{v}), \qquad \text{etc.}$$

This is called a *bilinear form.* A (real) inner product is a bilinear form satisfying additional conditions (symmetry and positivity).

Given a basis $\{\vec{d}_j\}$, let $Q_{jk} = \underline{Q}(\vec{d}_j, \vec{d}_k)$. Then

$$\underline{Q}(\vec{u}, \vec{v}) = Q_{jk} u^j v^k$$

for $\vec{u} \equiv u^j \vec{d}_j$, etc.

Definition: $\underline{Q}$ is *symmetric* if $\underline{Q}(\vec{u}, \vec{v}) = \underline{Q}(\vec{v}, \vec{u})$, or, equivalently, $Q_{jk} = Q_{kj}$. (Note that this equivalence does *not* require an ON basis, unlike that for symmetric *operators.* Indeed, "ON" is meaningless if there is no inner product.)

Definition: A *quadratic form* is $\underline{Q}(\vec{v}, \vec{v})$ for some bilinear $\underline{Q}$ — which may be assumed symmetric.

The bilinear form may be recovered from the quadratic form by calculating $\underline{Q}(\vec{u} + \vec{v}, \vec{u} + \vec{v})$ under the assumption of bilinearity and symmetry, and solving for $\underline{Q}(\vec{u}, \vec{v})$. (This trick of "polarization" has been used by us before.)

What happens to Q_{jk} under change of basis? Say $\vec{w}_k = R^j{}_k \vec{d}_j$.

$$\begin{aligned} {}^{\text{new}}Q_{k_1 k_2} &\equiv \underline{Q}(\vec{w}_{k_1}, \vec{w}_{k_2}) = R^{j_1}{}_{k_1} R^{j_2}{}_{k_2} \underline{Q}(\vec{d}_{j_1}, \vec{d}_{j_2}) \\ &= R^{j_1}{}_{k_1} R^{j_2}{}_{k_2} Q_{j_1 j_2} = (R^* Q R)_{k_1 k_2} . \end{aligned} \tag{\%}$$

Thus each index transforms like the index of a *co*vector ("covariantly").

Contrast the transformation law of the matrix of an *operator* ($\underline{A}\colon \mathcal{V} \to \mathcal{V}$):

$${}^{\text{new}}A^{k_1}{}_{k_2} = (R^{-1} A R)^{k_1}{}_{k_2} = (R^{-1})^{k_1}{}_{j_1} R^{j_2}{}_{k_2} A^{j_1}{}_{j_2} .$$

Here the column index is covariant, but the row index is "contravariant" (like the index of a vector in $\mathcal{V}$, not $\mathcal{V}^*$).

NOTE: If R is OG (e.g., if both bases are ON in an inner product space), then these transformation formulas are the same. If R is *not* OG, then (%) is *not* a "similarity transformation".

Theorem.

(A) *A symmetric bilinear form can be diagonalized by an orthogonal matrix.*

(B) *A symmetric bilinear form is represented, in* **some** *basis, by a diagonal matrix each of whose diagonal elements equals* 1, -1, *or* 0. *(The number of each of these is uniquely determined.)*

(C) *Two symmetric bilinear forms, at least one of which is positive definite, can be simultaneously diagonalized:*

$$Q_{(1)} = \underline{1}, \qquad Q_{(2)} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}.$$

Familiar example of (B): In two dimensions the possible normal forms (up to overall sign) correspond to the classification of conic sections.

$$\begin{array}{cccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \text{ellipse} & \text{hyperbola} & \text{parabola} & \text{straight line} \end{array}$$

Proof:

(A) Immediate from the preceding "note" and the spectral theorem for symmetric operators.

(B) Diagonalize as in (A). Then transform with

$$R = R^* = \begin{pmatrix} |\lambda_1|^{-\frac{1}{2}} & 0 & \\ 0 & |\lambda_2|^{-\frac{1}{2}} & \\ & & \ddots \end{pmatrix};$$

if $\lambda_j = 0$, replace $|\lambda_j|^{-\frac{1}{2}}$ by 1. (In more elementary terms, *rescale* the variables so as to replace $4x^2 - 2y^2$, for instance, with $x'^2 - y'^2$.) I omit the proof of uniqueness.

(C) Regard the positive definite form as an inner product, introduce an ON basis, and apply (A) to get a better ON basis. (The matrix of the inner product is $\underline{1}$ in any ON basis.)

Observation 1.

(A) *A bilinear form* $\underline{Q}\colon \mathcal{V}\times\mathcal{V}\to \mathbf{R}$ *defines a linear operator* $\underline{\tilde{Q}}\colon \mathcal{V}\to\mathcal{V}^*$ *by*

$$\underline{\tilde{Q}}_{\vec{u}}(\vec{v}) \equiv \underline{Q}(\vec{u},\vec{v}).$$

(When $\underline{Q}$ *is the inner product,* $\underline{\tilde{Q}} = \underline{G}$*.) Conversely, such an operator defines a form.*

(B) *An operator* $\underline{A}\colon \mathcal{V}\to\mathcal{V}$ *defines a bilinear function* $\underline{A}\colon \mathcal{V}^*\times\mathcal{V}\to\mathbf{R}$ *by*

$$\underline{A}(\tilde{U},\vec{v}) \equiv \tilde{U}(\underline{A}\vec{v}).$$

Conversely, such a bilinear function defines an operator (since $\mathcal{V}^{**}=\mathcal{V}$*). [If* $\mathcal{V}$ *is a real inner-product space, an operator* $\underline{A}$ *defines a bilinear form* $\underline{A}\colon\mathcal{V}\times\mathcal{V}\to\mathbf{R}$ *by* $\underline{A}(\vec{u},\vec{v})\equiv\langle\vec{u},\underline{A}\vec{v}\rangle$*.]*

(C) *In each case, the matrix of the operator and that of the associated bilinear form are the same. The correspondence between operators and bilinears is an isomorphism.*

Observation 2. *Each pair of vectors* $\vec{v}_1$, $\vec{v}_2\in\mathcal{V}$ *defines a bilinear form* $\vec{v}_1\otimes\vec{v}_2$ **on** $\mathcal{V}^*$ *by*

$$(\vec{v}_1\otimes\vec{v}_2)(\tilde{U}^1,\tilde{U}^2)\equiv\tilde{U}^1(\vec{v}_1)\,\tilde{U}^2(\vec{v}_2),\quad \forall\tilde{U}_1,\tilde{U}_2\in\mathcal{V}^*.$$

The matrix of $\vec{v}_1\otimes\vec{v}_2$ *is* $\{v_1^j\,v_2^k\}$*. The object* $\vec{v}_1\otimes\vec{v}_2$ *depends bilinearly on* $\vec{v}_1$ *and* $\vec{v}_2$ *.*

Similarly, each pair $\tilde{U}^1$, $\tilde{U}^2\in\mathcal{V}^*$ *defines a bilinear form* $\tilde{U}^1\otimes\tilde{U}^2$ *on* $\mathcal{V}$ *by*

$$(\tilde{U}^1\otimes\tilde{U}^2)(\vec{v}_1,\vec{v}_2)\equiv\tilde{U}^1(\vec{v}_1)\,\tilde{U}^2(\vec{v}_2).$$

Its matrix is $\{U^1_j\,U^2_k\}$ *and it depends bilinearly on* $\tilde{U}^1$ *and* $\tilde{U}^2$*.*

The operation $\otimes$ is called *tensor product.*