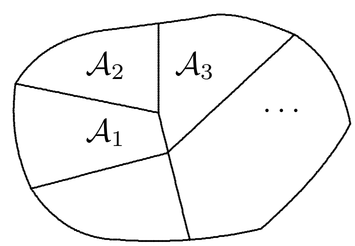
Factor spaces *(Sec. 11)*

Partitions and equivalence relations *(See Sec. 2)*

Definition: A *partition* of a set $\mathcal{A}$ is a way of representing $\mathcal{A}$ as a union of disjoint sets:

$$\mathcal{A} = \underbrace{\mathcal{A}_1 \cup \mathcal{A}_2 \cdots}_{\text{possibly uncountable}} \equiv \bigcup_{j\in J} \mathcal{A}_j\,, \qquad \mathcal{A}_j \cap \mathcal{A}_k = \emptyset \quad \text{if } j \neq k.$$



Definition: An *equivalence relation* "$\simeq$" on $\mathcal{A}$ is a relation [a subset of $\mathcal{A} \times \mathcal{A}$, or a "property" of two objects in $\mathcal{A}$] which is

1) *reflexive*: $x \simeq x \quad (\forall x \in \mathcal{A})$,

2) *symmetric*: $x \simeq y \;\Rightarrow\; y \simeq x$.

3) *transitive*: $x \simeq y$ and $y \simeq z \;\Rightarrow\; x \simeq z$.

Examples

1. Equality (of numbers, say)
2. Congruence of triangles
3. Similarity of triangles
4. Similarity of matrices: $x = R^{-1}yR$
5. $f(x) = f(y)$ (for a fixed function f)

Theorem. *Every equivalence relation defines a partition, and vice versa.*

Proof:

$\Leftarrow$: Given a partition, call x and y equivalent ($x \simeq y$) if x and y belong to the same component $\mathcal{A}_j$.

$\Rightarrow$: Given an equivalence relation, define $\overline{x}$ (also denoted $[x]$) to be the set of elements equivalent to x:

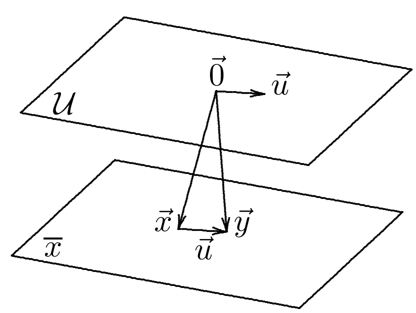
$$\overline{x} = \{\, y \in \mathcal{A} : x \simeq y \,\}.$$

Then $\overline{x} = \overline{y}$ iff $x \simeq y$. If $x \not\simeq y$, then $\overline{x} \cap \overline{y} = \emptyset$ (else transitivity would be violated). Reflexivity $\Rightarrow$ $x \in \overline{x}$. Thus $\mathcal{A}$ is the disjoint union of the sets $\overline{x}$ (called *equivalence classes*).

Looking back at the examples, we can see how each of them divides the set of objects under consideration into equivalence classes. In the case of triangles, the first three examples constitute increasingly coarse classifications; at each step the previous equivalence classes are combined into larger classes.

Definition of a factor space

Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Given $\vec{x} \in \mathcal{V}$, define $\overline{x} \equiv \{\, \vec{x} + \vec{u} : \vec{u} \in \mathcal{U} \,\}$. Equivalently, define $\vec{x} \simeq \vec{y}$ to mean that $\vec{x} - \vec{y} \in \mathcal{U}$, and let $\{\overline{x}\}$ be the corresponding collection of equivalence classes. This partition is called the *factor space* (or *quotient space*) $\mathcal{V}/\mathcal{U}$. [“/” is pronounced “mod” or “modulo”.]



Note: $\vec{x} \in \mathcal{U} \iff \overline{x} = \mathcal{U} = \overline{\vec{0}}$.

Geometrically, subspaces are lines, planes, ... through the origin. Elements of factor spaces are lines, planes, ... that *don't* pass through the origin; elements of $\mathcal{V}/\mathcal{U}$ are *parallel to* $\mathcal{U}$.

The elements of $\mathcal{V}/\mathcal{U}$ are called *cosets* or *affine subspaces* (as well as “equivalence classes”).

Let $f(\vec{x})$ be a function on $\mathcal{V}$ such that

$$\vec{x} \simeq \vec{y} \;\Rightarrow\; f(\vec{x}) = f(\vec{y}).$$

Thus f is constant on each coset. We can identify f with a function (also called f) on $\mathcal{V}/\mathcal{U}$, defined by:

$$\alpha \in \mathcal{V}/\mathcal{U} \;\Rightarrow\; f(\alpha) \equiv f(\vec{x}) \text{ for any } \vec{x} \in \alpha$$

or, more briefly,

$$f(\overline{x}) \equiv f(\vec{x}) \qquad (\forall \overline{x} \in \mathcal{V}/\mathcal{U}). \tag{‡}$$

($\vec{x}$ is called a *representative* of $\overline{x}$.) We say that f *lifts* to a function on $\mathcal{V}/\mathcal{U}$.

REMARK: In (‡) (and elsewhere later) we're using $\overline{x}$ as a *variable* for an arbitrary element of $\mathcal{V}/\mathcal{U}$; then we use x to stand for an arbitrary element of $\overline{x}$. This is OK as long as you don't forget that $\overline{x} = \overline{y}$ for many $\vec{y}$'s not equal to $\vec{x}$. When we consider a coset and call it $\overline{x}$, we are not necessarily committed to *any particular member of the coset* to be called "x". Whenever this somewhat sloppy notation would cause a danger of confusion, it is better to call the coset α (say) instead of $\overline{x}$.

Now consider $g\colon \mathcal{V} \to \mathcal{V}$ defined by $g(\vec{x}) \equiv 3\vec{x}$. (The 3 could be any scalar; we are considering a particular one only for concreteness.) This function lifts, *if* we take a quotient at the *codomain* end of the mapping, too:

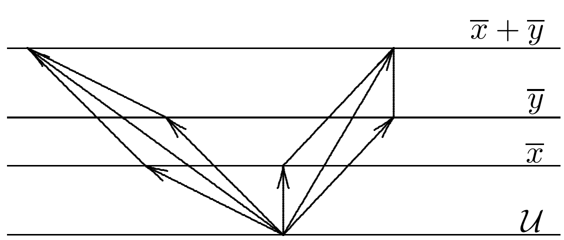
$$3\overline{x} \equiv \overline{(3\vec{x})}.$$

(In the picture, $3\overline{x}$ is the plane parallel to $\overline{x}$ but 3 times farther from $\mathcal{U}$.)

Similarly, we can define the *sum* of two cosets by

$$\overline{x} + \overline{y} = \overline{\vec{x} + \vec{y}}.$$

Otherwise stated: Pick an arbitrary $\vec{x} \in \overline{x}$ and an arbitrary $\vec{y} \in \overline{y}$. Find the coset containing the vector $\vec{x} + \vec{y}$. This coset will be independent of which $\vec{x}$ and $\vec{y}$ you chose, and it is defined to be $\overline{x} + \overline{y}$.



Theorem 11.2. *$\mathcal{V}/\mathcal{U}$ is a vector space under addition and scalar multiplication as defined above. The zero vector in this space is $\overline{0} \equiv \overline{\vec{0}} = \mathcal{U}$.*

Proof: left to homework (along with the details of the argument that the vector operations do indeed lift).

The standard application

Let $\underline{A}\colon \mathcal{V} \to \mathcal{W}$ be a linear operator. Then $\ker \underline{A} =$ (solution space of homogeneous equation $\underline{A}\vec{v} = \vec{0}$) is a subspace of $\mathcal{V}$. *For fixed $\vec{b} \in \operatorname{ran} \underline{A} \subseteq \mathcal{W}$, the solution space of the inhomogeneous equation $\underline{A}\vec{v} = \vec{b}$ is a coset in $\mathcal{V}$ parallel to* $\ker \underline{A}$; i.e., a member of $\mathcal{V}/\ker \underline{A}$. [Proof: $\underline{A}\vec{v}_1 = \vec{b}$ and $\underline{A}\vec{v}_2 = \vec{b} \;\Rightarrow\; \underline{A}(\vec{v}_1 - \vec{v}_2) = \vec{0}$, so all solutions belong to the same coset. Conversely, $\underline{A}\vec{u} = \vec{0}$ and $\underline{A}\vec{v}_1 = \vec{b} \;\Rightarrow\; \underline{A}(\vec{v}_1 + \vec{u}) = \vec{b}$, so the entire coset containing a solution consists of solutions.] *The solution set of $\underline{A}\vec{v} = \vec{b}_1 + \vec{b}_2$ is the factor-space sum of the solution sets for $\vec{b}_1$ and $\vec{b}_2$ ("superposition principle"). Simlarly for scalar multiples.*

Example: In calculus, the *indefinite integral* of f is the space of solutions of the inhomogeneous linear equation $dy/dx = f(x)$. An indefinite integral is a vector in the factor space

$$\text{(continuous functions)}/\text{(constant functions)}$$

(the constants being the kernel of d/dx). In integral tables each coset is represented by an arbitrary element in it:

$$\int \cos x \, dx = \sin x.$$

But in textbooks we write

$$\int \cos x \, dx = \sin x + C$$

as a reminder that we really mean the whole coset. We calculate the integral of a sum by adding cosets:

$$\int (x + \cos x) \, dx = \frac{x^2}{2} + \sin x + C$$

[*not* $\frac{x^2}{2} + C_1 + \sin x + C_2$, since we're adding cosets, not functions].

Factor spaces in everyday life *(More examples)*

1. Equivalence classes of square-integrable functions *(a step toward $\mathcal{L}^p$ spaces)*

This has already been discussed to some extent (and will be a central ingredient in Math. 641–642).

Let $X = (a,b) \subseteq \mathbf{R}$. (For that matter, X could be *any* set on which an integral can be defined.) Let

$$\mathbf{L}^2(X) = \{\, f\colon X \to \mathcal{F} : \int_X |f(x)|^2\,dx < \infty \,\}.$$

This is a vector space. A natural candidate for inner product is

$$f \cdot g \equiv \int_X f(x)\,\overline{g(x)}\,dx.$$

However, it doesn't satisfy the positive-definiteness condition. We can cure this by "identifying" any two functions whose difference has norm zero (e.g., two functions that coincide except on a finite set): Let

$$\mathcal{I} \equiv \{\, f \in \mathbf{L}^2(X) : \int_X |f(x)|^2\,dx = 0 \,\}$$

and let

$$\mathcal{L}^2(X) \equiv \mathbf{L}^2(X)/\mathcal{I}.$$

Then $f \cdot g$ lifts to an inner product on $\mathcal{L}^2$, and the latter is what we mean when we talk about the Hilbert space of "square-integrable functions".

An $\overline{f} \in \mathcal{L}^2$ contains at most one continuous representative (since a nonzero continuous function can't belong to $\mathcal{I}$). If $\overline{f}$ contains *no* continuous function, it has no "canonical" representative. But if we're studying solutions of differential equations, the interesting functions usually *will* be continuous, and the temptation to smudge the distinction between $\overline{f}$ and f will be irresistible.

2. Equivalence classes of singular functions *(a step toward renormalization theory for quantized fields)*

A. Consider a function f such that $|f(x)| \to \infty$ as $x \to 0$. For example,

$$f(x) = \frac{e^x}{x} = \frac{\cosh x}{x} + \frac{\sinh x}{x}\,. \qquad (*)$$

Can we divide f into its "singular part" plus its "smooth part"? No, $(*)$ shows that this idea is ambiguous. In fact, *any* smooth function whatsoever could be separated out:

$$f(x) = \underbrace{f(x) - s(x)}_{\text{singular}} + \underbrace{s(x)}_{\text{smooth}} \qquad \text{for any smooth } s.$$

The point is that a singular function plus a smooth function is still singular. Therefore, the "singular part" of f is defined only modulo smooth functions.

Still, it makes sense to observe that $f(x) \sim \frac{1}{x}$ as $x \to 0$ (rather than $\sim \frac{2}{x}$ or $\sim \frac{1}{x^2}$, say). Indeed, f has the Laurent series

$$\frac{e^x}{x} \sim \frac{1}{x} + 1 + \frac{x}{2} + \cdots .$$

Extracting the singular term(s) of such a series can be regarded as a calculation of f's coset in the space

(possibly singular functions)/(smooth functions).

(This is still rather vague, because I have not specified a domain for the functions nor specified how pathological the elements of the space of singular functions are allowed to be. The essential algebraic point is independent of such technical analytical details.)

Earlier we saw that many a coset in $\mathcal{L}^2$ has a "natural" representative — its unique continuous member. In the present situation, similarly, whenever f has a Laurent series there is a natural convention for defining its singular part; in our example, it is

$$f_{\text{sing}}(x) \equiv \frac{1}{x}, \qquad f_{\text{sm}}(x) = \frac{e^x - 1}{x}.$$

B. Suppose, however, that $f(x) \sim \ln x$ as $x \to 0$. Consider, for example,

$$f(x) = \ln \frac{x}{y - x}.$$

For fixed $y > 0$, this is defined for $0 < x < y$. Note that

$$f(x) = \ln \frac{\frac{x}{y}}{1 - \frac{x}{y}} = \ln \frac{x}{y} - \ln\left(1 - \frac{x}{y}\right).$$

Now use the fact that

$$\ln(1 + z) \sim z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots \quad \text{as } z \to 0.$$

You get

$$f(x) \sim \ln \frac{x}{y} + \frac{x}{y} + \frac{1}{2}\frac{x^2}{y^2} + \frac{1}{3}\frac{x^3}{y^3} + \cdots .$$

It's tempting to say

$$f_{\text{sing}} \equiv \ln \frac{x}{y}, \qquad f_{\text{sm}} = \frac{x}{y} + \cdots .$$

But this is too arbitrary to be justified. Note that

$$\begin{aligned} \ln \frac{x}{y} &= \ln\left(cx \frac{1}{cy}\right) \\ &= \ln cx - \ln cy \quad \text{for arbitrary } c > 0, \end{aligned}$$

and the $-\ln cy$ is a nonsingular term (as a function of x). So we could equally well write

$$f_{\text{sing}} \equiv \ln cx, \qquad f_{\text{sm}} = -\ln cy + \frac{x}{y} + \frac{1}{2}\frac{x^2}{y^2} + \cdots . \tag{\dagger}$$

This doesn't essentially change what we mean by f_{sing} — it's still the same coset. But our formula for f_{sm} contains an inherently arbitrary constant, c. [It's no good saying that $c = 1$ is the "natural" choice. In a physical application, x and y will usually have physical dimensions, say length, and $1/c$ will be a quantity of the same type. If we set $c = 1$ and then change the unit of length from inches to feet, suddenly c becomes 12!]

In modern theories of fundamental physics, certain constants of nature, such as the masses of elementary particles and the ranges of their interactions, can arise by mechanisms of this type. These constants are not determined by the theory one starts with; they must be measured experimentally. (This is called "dimensional transmutation", because a constant without physical dimensions in the original theory can be replaced by a length or a mass in the final theory, which has been "made finite", or "renormalized".) The notorious "renormalization" or "subtraction of infinity" in quantum field theory is basically an operation of *subtracting the singular part of a function to obtain a smooth remainder*, and it is afflicted by the ambiguities we have seen to be inherent in such a calculation.

C. This application has another aspect, which stands the mathematics of the previous discussion on its head, so to speak. Frequently in a quantum-field-theoretic calculation, the singular terms in the expansion of a function are especially simple in their dependence on some other variable in the problem. This is modeled in our example, (†), in the following way: Note that f_{sing} is independent of y, and that changing the arbitrary constant c modifies f_{sm} only by adding a constant (since $\ln c'y = \ln cy + \ln(c'/c)$). It is therefore possible to say with a minimum of ambiguity what is meant by "the smooth part of f". Although f_{sm} is not well-defined as a function, it *is* well-defined as a member of the factor space

$$\text{(smooth functions)}/\text{(smooth functions independent of } y).$$

(Recall, in contrast, that when we started this discussion we concluded that the smooth part of the function in $(*)$ was totally ill-defined, unless we introduced an arbitrary convention involving Laurent series.) This is really the crux of renormalization theory. In a physical calculation, one is really interested in defining f_{sm}. The embarrasing "infinity", f_{sing}, fortunately turns out to be so simple in its dependence on the physical variables of the problem that it can be regarded as physically trivial. This is what justifies ignoring it, even though it is infinite! The only thing that remains of the singular terms in the final result of a calculation involving renormalization is an *ambiguous smooth* term, reflecting the fact that all that the theory determines is the *coset* of the answer modulo the subspace of "physically trivial" functions.