Math. 640 December 2018

Final Examination – Solutions

1. *(15 pts.)* From general principles (without extensive calculations): Which of these matrices can be diagonalized by a similarity transformation by a unitary matrix? Which can be diagonalized, but only by a nonunitary matrix? Which cannot be diagonalized at all? (The field of scalars is the complex numbers.)

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & i \\ -i & 0 \end{pmatrix}.$$

A is not Hermitian , so it can't be diagonalized by a unitary. To check that it can be diagonalized by a nonunitary, we need to do a *little* bit of calculation: The characteristic equation is

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5.$$

Its discriminant is $(-6)^2 - 4\cdot 5 = 16 \neq 0$, so it has distinct roots. So the Jordan form is diagonal.

B is "essentially" in a nondiagonal Jordan form. (To put it into standard form, interchange the two basis vectors. Or observe that its characteristic equation *does* have a double root but it's not a multiple of the identity matrix.) So it can't be diagonalized.

C is Hermitian, so it can be diagonalized by a unitary.

2. *(30 pts.)* The matrix $M = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has only two eigenvalues, $\lambda_1 = 4$ and $\lambda_2 = 2$.

Find the orthogonal spectral projections onto the two eigenspaces.

Method 1: Use Sylvester's formula.

$$P_1 = \frac{\lambda_2 - M}{\lambda_2 - \lambda_1} = \frac{1}{2}\begin{pmatrix} 3-2 & 1 & 0 \\ 1 & 3-2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$P_2 = \frac{\lambda_1 - M}{\lambda 1 - \lambda 2} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Method 2: Find the eigenvectors.

Eigenvectors for $\lambda = 4$*:* Insert this value of λ into the matrix $M - \lambda$ and reduce:

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $z = 0$ and $x = y$. The general vector is $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$. A normalized eigenvector is $\hat{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

To project a vector $\vec{w}$ onto $\hat{u}_1$ we multiply $\hat{u}_1$ by the inner product $\hat{u}_1 \cdot \vec{w}$. This operation is represented by the matrix whose jk entry is the product of the j th and k th elements of $\hat{u}_1$ (see B&W (28.16)):

$$P_1 = \hat{u}_1 \otimes \hat{u}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Eigenvectors for $\lambda = 2$ *:*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus z is arbitrary and $x + y = 0$. The most general eigenvector is of the form $\begin{pmatrix} x \\ -x \\ z \end{pmatrix}$. An orthonormal basis for this eigenspace is

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Then P_2 is the sum of the projections onto each of these:

$$P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which agrees with what Sylvester gave us.

3. *(25 pts.)* Explain how the concepts of *coset* and *factor space* are relevant to the solving of inhomogeneous linear differential equations, such as

$$\frac{d^2y}{dt^2} + 25y = \cos 6t + 25e^{-2t} + \frac{\sin 2t}{t^2 + 1}.$$

(You are not required to solve this equation. Discuss it as a concrete example if you like.)

First, any two solutions of the same inhomogeneous equation differ by a solution of the corresponding homogeneous equation, so the solutions form a coset in the factor space $\mathcal{V}/\ker \underline{L}$, where $\underline{L}$ is the linear operator involved and $\mathcal{V}$ is its domain.

Second, when the inhomogeneous part of the equation is a sum of terms, $\vec{b}_1 + \vec{b}_2 + \cdots$, we can construct a solution by finding a solution for each equation $\underline{L}\,\vec{b}_j = 0$ and adding up. Here we are really doing addition in the factor space; we drop back to the original space at the end by adding the general solution of the homogeneous equation.

4. *(40 pts.)*

(a) Find the Jordan canonical form of $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, and exhibit a corresponding Jordan basis (i.e., the generalization of an eigenbasis).

$\det(A - \lambda) = (3-\lambda)^3$ implies that $\lambda = 3$ is the only eigenvalue. The eigenvectors are the kernel of $N \equiv \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, and so it's easy to see that $\vec v_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only independent eigenvector.

The Jordan form must therefore be $J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. The second basis vector should satisfy

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

whence $z = 0$, $y = \frac{1}{2}$, x arbitrary. Let's pick $x = 0$: $\vec v_2 \equiv \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$. The third vector then should satisfy

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix},$$

or $z = \frac{1}{4}$, $y = -\frac{1}{8}$; $\vec v_3 \equiv \begin{pmatrix} 0 \\ -\frac{1}{8} \\ \frac{1}{4} \end{pmatrix}$. Then $S \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$ is the matrix mapping coordinates from the Jordan basis to the original basis.

(b) Solve the ODE system $\dfrac{d\vec x}{dt} = A\vec x(t)$ with initial data $\vec x(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Method 1: Since the matrix is already upper triangular, we could simply solve the system from the bottom up. But that's no fun. ...

Method 2: $e^{At} = e^{3t}[1 + Nt + \frac{1}{2}N^2t^2]$, where $N^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus

$$e^{At} = e^{3t} \begin{pmatrix} 1 & 2t & 2t^2 + t \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}.$$

Apply this matrix to the given initial vector to get $\vec x(t) = \begin{pmatrix} 2te^{3t} \\ e^{3t} \\ 0 \end{pmatrix}$.

Method 3: A basis of solutions to the system is

$$\vec{x}_1(t) \equiv e^{3t}\vec{v}_1\,, \qquad \vec{x}_2(t) \equiv e^{3t}\vec{v}_2 + t\,e^{3t}\vec{v}_1\,,$$

and a third one which contains a term involving $\vec{v}_3$ and hence won't be needed to match our initial data. The initial data for these solutions are $\vec{x}_1(0) = \vec{v}_1$, $\vec{x}_2(0) = \vec{v}_2$, etc. Since $\vec{x}(0) = 2\vec{v}_2$, the desired solution is

$$2\vec{x}_2(t) = \begin{pmatrix} 2te^{3t} \\ e^{3t} \\ 0 \end{pmatrix}.$$

Method 4: Apply the inverse of S to $\vec{x}(0)$ to get the initial data in the Jordan coordinates. Solve the system $\vec{u}\,' = J\vec{u}$ with that data vector and apply S to the result. (But that isn't fun either.)

5. *(40 pts.)*

(a) Let $\mathcal{V}$ and $\mathcal{U}$ be inner product spaces, and let $\underset{\sim}{A}:\mathcal{V}\to\mathcal{U}$ be a linear operator with adjoint operator $\underset{\sim}{A}^*$. Prove that

$$\ker \underset{\sim}{A} = \ker(\underset{\sim}{A}^*\underset{\sim}{A}) \qquad \text{and} \qquad \ker \underset{\sim}{A}^* = \ker(\underset{\sim}{A}\,\underset{\sim}{A}^*).$$

(You may assume that the spaces are finite-dimensional, or at least that $\underset{\sim}{A}$ and $\underset{\sim}{A}^*$ are everywhere defined.)

$\underset{\sim}{A}\,\vec{v} = \vec{0} \;\Rightarrow\; \underset{\sim}{A}^*\underset{\sim}{A}\,\vec{v} = \vec{0}$, so $\ker \underset{\sim}{A} \subset \ker(\underset{\sim}{A}^*\underset{\sim}{A})$. Conversely,

$$\underset{\sim}{A}^*\underset{\sim}{A}\,\vec{v} = \vec{0} \;\Rightarrow\; 0 = \vec{v}\cdot\,\underset{\sim}{A}^*\underset{\sim}{A}\,\vec{v} = \|\,\underset{\sim}{A}\,\vec{v}\|^2 \;\Rightarrow\; \underset{\sim}{A}\,\vec{v} = \vec{0}.$$

The second equality is proved in the same way.

(b) Making a nonrigorous extrapolation from (a) (i.e., ignoring the "everywhere defined" technicality), explain why you might expect the following to be true:

Let Ω be a bounded, connected region in $\mathbf{R}^n$. Then in Ω the only solutions of Laplace's equation, $\nabla^2\phi = 0$, whose normal derivatives vanish on the boundary of Ω are the constant functions. *Hints:* $\underset{\sim}{A}$ is the gradient operator,

$$\underset{\sim}{A}\,\phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \ldots\right).$$

The inner products are the standard ones with $\|\phi\|^2 = \int_\Omega |\phi|^2\,d^nx$ for scalar functions, $\|\vec{V}\|^2 = \int_\Omega \vec{V}\cdot\vec{V}\,d^nx$ for (real-valued) vector functions. Consider only vector functions whose normal components vanish on the boundary.

$\underset{\sim}{A}$ maps scalar functions to vector functions. Therefore, $\underset{\sim}{A}^*$ maps vector functions to scalar functions. What is $\underset{\sim}{A}^*$? It must satisfy

$$\langle\,\underset{\sim}{A}^*\vec{V}, \phi\rangle = \langle\vec{V},\,\underset{\sim}{A}\,\phi\rangle = \int_\Omega \sum_{j=1}^n V_j\frac{\partial\phi}{\partial x_j}.$$

Integrate each term by parts, noting that there is no boundary contribution:

$$\langle \underset{\sim}{A}^*\vec{V}, \phi\rangle = -\int_\Omega \sum_{j=1}^{n} \frac{\partial V_j}{\partial x_j}\phi = -\int_\Omega (\nabla\cdot\vec{V})\phi \quad \text{for all } \phi\,.$$

Thus $\underset{\sim}{A}^*$ is the negative of the divergence operator, and $\underset{\sim}{A}^*\underset{\sim}{A} = -\nabla^2$. According to (a), the kernel of ∇^2 is just the kernel of the gradient operator; since Ω is connected, the only functions with all partial derivatives zero are the constants.

6. *(30 pts.)* Let $\{\vec{d}_j\}$ and $\{\vec{e}_k\}$ be two bases for a vector space $\mathcal{V}$, related by

$$\vec{e}_k = \sum_{j=1}^{N} S^j{}_k\, \vec{d}_j.$$

(a) What is the definition of "dual basis"?

If $\{\vec{d}_j\}$ is a basis for $\mathcal{V}$, then the dual basis $\{\tilde{D}^j\}$ is the (unique) basis of $\mathcal{V}^*$ such that $\tilde{D}^j(\vec{d}_k) = \delta^j_k$ for all j and k. In other words, $\tilde{D}^j$ "picks out" the jth coordinate (with respect to the d basis) of whatever vector in $\mathcal{V}$ it acts upon.

(b) Let $\{\tilde{D}^j\}$ and $\{\tilde{E}^k\}$ be the dual bases to $\{\vec{d}_j\}$ and $\{\vec{e}_k\}$. A linear functional $\tilde{U}\in\mathcal{V}^*$ is represented with respect to the dual bases as

$$\tilde{U} = \sum_{j=1}^{N} \mu_j\, \tilde{D}^j = \sum_{k=1}^{N} \nu_k\, \tilde{E}^k.$$

What is the transformation law (i.e., the formula for ν_k in terms of μ_j)?

Let $\{\nu_k\}$ be the new coordinates: $\tilde{U} = \sum_{k=1}^{N} \nu_k \tilde{E}^k$. Then $\nu_k = \sum_{j=1}^{N} S^j{}_k\, \mu_j$.

(c) Let M be a linear mapping from $\mathcal{V}^*$ into $\mathcal{V}$. By what kind of matrix would you represent M? (Which indices would you write "up" or "down"?)

The matrix has two indices up: $(M\tilde{U})^j = \sum_{k=1}^{N} M^{jk}\mu_k$.

(d) How does the matrix in (c) transform when the basis is changed?

When we transform to the $\vec{e}$ basis, each index is acted upon by the matrix S^{-1}. That is, the new matrix elements are

$$\sum_{l,m} (S^{-1})^j{}_l\, (S^{-1})^k{}_m\, M^{lm}$$

— or, if you prefer a more systematic notation where j always goes with $\vec{d}$ and k always goes with $\vec{e}$,

$$\sum_{j_1,j_2} (S^{-1})^{k_1}{}_{j_1}\, (S^{-1})^{k_2}{}_{j_2}\, M^{j_1 j_2}\,.$$

7. *(20 pts.)* Given two linearly independent vectors, $\vec{u}$ and $\vec{v}$, in $\mathbf{R}^3$, how would you find the matrix of a rotation through an angle θ in the plane they span? (Treat this as more of an essay question than a calculational one, though you might want to work out an example.) *Hints:* You will probably want to mention these:
   - the cross product $\vec{u} \times \vec{v}$,
   - the differential equation $\frac{dR}{dt} = AR$, where A is a certain antisymmetric matrix,
   - the exponential of a matrix.

$\vec{w} \equiv \vec{u} \times \vec{v}$ is perpendicular to the plane and must point along the axis of the rotation. The antisymmetric matrix whose nonzero elements are the components of $\vec{w}$,

$$A = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} = \pm(\vec{u} \otimes \vec{v} - \vec{v} \otimes \vec{u})$$

is the generator of the rotations via $\frac{dR}{dt} = AR$ with initial condition $R(0) = I$. Thus $R(t) = e^{tA}$. There are two loose ends: (1) Are the angles normalized so that $\theta = t$, or is there a scale factor? (2) Which direction are we rotating? These are most easily checked by looking at a special case: $\vec{u} = s\hat{\imath}$, $\vec{v} = \hat{\jmath}$. Then $\vec{w} = s\hat{k}$,

$$A = \begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{tA} = \begin{pmatrix} 0 & \sin(ts) & 0 \\ -\sin(ts) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the rotation angle is $\theta = ts$; in general, the role of s will be played by $\|\vec{w}\|$ and the sign is more cumbersome to discuss but can be figured out *a posteriori* in any particular case.