Math. 640 Fall, 2018

Homework 11, due November 14

1. Operator-valued power series were introduced in lecture without a rigorous discussion of the meaning of such series. Show that these two definitions of *convergence of a sequence* (hence *of a series*) of operators are equivalent:

$$\underline{A}_n \to \underline{A} \quad \text{means either of:}$$

   (A) In some fixed basis, the matrices A_n converge to the matrix A (in the usual topology of $\mathbf{C}^{N^2}$, $\quad N \equiv \dim \mathcal{V}$).

   (B) For all $\vec{v} \in \mathcal{V}$, $\quad \underline{A}_n\vec{v} \to \underline{A}\vec{v}$ (in the usual topology of $\mathcal{V} \cong \mathbf{C}^N$).

   HINT: Recall that any norm on $\mathbf{R}^n$ defines the "usual" topology. This principle extends to $\mathbf{C}^n$.

2. Show that if $\underline{A}$ is diagonalizable, then the definition of $f(\underline{A})$ via Cauchy's formula agrees with the definition by diagonalization. HINT: First consider the action of $f(\underline{A})$ on an eigenvector; then extend by linearity.

3. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, and let $f(x) = e^{3x}$. Calculate $f(A)$ in two ways:

   (A) from the formula (#);

   (B) from the power series $e^{3A} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(3A)^n$.

   (Verify that the answers are consistent.)

4. Show that the solution of the *second-order* equation

$$\frac{d^2\vec{x}}{dt^2} = -\underline{A}\vec{x}; \qquad \vec{x}(0) \text{ and } \vec{x}\,'(0) \equiv \left.\frac{d\vec{x}}{dt}\right|_{t=0} \text{ given,}$$

   can be written in terms of trigonometric functions of the operator $\sqrt{\underline{A}}$. (Assume that $\underline{A}$ is [Hermitian and] positive definite, so that $\sqrt{\underline{A}}$ is well-defined.)