Math. 640 Fall, 2018

Homework 12, due November 28

1. Recall that $\underline{A}$ is *positive* if $\vec{v} \cdot \underline{A}\vec{v} \geq 0, \quad \forall \vec{v}$.

   (a) Prove that every positive operator in a [finite-dimensional] complex inner-product space is Hermitian.

   (b) Give a counterexample for real inner-product spaces. HINT: Find a nonsymmetric 2×2 matrix A for which $\vec{v} \cdot A\vec{v}$ is a square for all $\vec{v} \in \mathbf{R}^2$.

2. Prove the polar decomposition theorem for the case where $\underline{A}$ is not invertible:

   (a) Show that $\ker(\underline{A}^* \underline{A}) = \ker \underline{A}$.

   (b) Use this fact to construct a suitable $\underline{U}$ for this case. ($\underline{U}$ will not be unique.) HINT: Define $\underline{U}$ on $(\ker \underline{A})^\perp$ (using the spectral theorem to construct a substitute for $(\underline{A}^* \underline{A})^{-\frac{1}{2}}$) and on $\ker A$ separately.

3. Why is Sylvester's formula not correct when $\underline{A}$ has a nondiagonal Jordan canonical form? HINT: Use formula (#).