Math. 640 Fall, 2018

Homework 2, due September 12

1. *[Bowen & Wang, p. 45]* Let $\mathcal{V}$ be a vector space and consider the set $\mathcal{V} \times \mathcal{V}$. Define addition in $\mathcal{V} \times \mathcal{V}$ by
$$(\vec{u}, \vec{v}) + (\vec{x}, \vec{y}) = (\vec{u} + \vec{x}, \vec{v} + \vec{y})$$
and multiplication by complex numbers by
$$(\lambda + i\mu)(\vec{u}, \vec{v}) = (\lambda\vec{u} - \mu\vec{v}, \mu\vec{u} + \lambda\vec{v})$$
where $\lambda, \mu \in \mathbf{R}$. Show that $\mathcal{V} \times \mathcal{V}$ is a vector space over the field of complex numbers.

2. *[cf. Milne, p. 21]* Which of the following sets of triples of real numbers (x, y, z) are subspaces of $\mathbf{R}^3$?
   (a) $x = 2 - y$; y and z arbitrary.
   (b) $z = 2y - x$; x and y arbitrary.
   (c) $z = |x|$; x and y arbitrary.
   (d) x and $y \geq 0$; z arbitrary.
   (e) $x = y = 2z$; z arbitrary.
   (f) $x = 3y$; $z = 0$; y arbitrary.

3. *[Bowen & Wang p. 54]* Are the complex numbers $2 + 4i$ and $6 + 2i$ linearly independent with respect to the field of real numbers, $\mathbf{R}$? Are they linearly independent with respect to the field of complex numbers?

4. *[Milne p. 27]* Consider the vector space of all real-valued continuous functions defined on $(-\infty, \infty)$. Which of the following conditions define a subspace?
   (a) $x(1) = 0$.
   (b) $x(1) + x(-1) = 0$.
   (c) $x(1) + x(2) = 2$.
   (d) $\int_{-1}^{1} x(t)\, dt = 0$.
   (e) x is odd: $x(-t) = -x(t)$.
   (f) x is periodic with period 2π.

5. Consider the following three sets of vectors in $\mathbf{C}^3$.

   (a) {(1,0,0), (0,-1,0), (1,1,0)}

   (b) {(1,0,1), (2,2,2)}

   (c) {(1,0,0), (0,-1,0), (0,-1,1)}

   Which of these is (are) linearly independent? Which span(s) $\mathbf{C}^3$? Which is (are) a basis for $\mathbf{C}^3$? Which could be made into a basis by adding another vector?

6. *[Bowen & Wang p. 54]* Let $\mathcal{M}^{2\times 2}$ denote the vector space of all 2×2 matrices with elements from the real numbers $\mathbf{R}$. Is either of the following sets a basis for $\mathcal{M}^{2\times 2}$?

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 8 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

7. *[Milne p. 34]* Find a minimal spanning set (hence a basis) for span $\mathcal{S}$ when

   (a) $\mathcal{S} = \{(1,0,2,0), (0,2,0,1), (1,2,2,1), (-1,0,3,0)\}$

   (b) $\mathcal{S} = \{(0,1,2,3), (3,0,1,2), (2,3,0,1), (1,2,3,0)\}$

   (c) $\mathcal{S} = \{1+2t, 1-t+t^2, 4+t^2, 1-t^3, 4+2t+t^2+t^3\}$

8. *[cf. Milne pp. 34–35]* Prove that every system of n homogeneous linear equations in $n+1$ or more unknowns has a nonzero solution. *Hint:* Think of the columns of the coefficient matrix as vectors in $\mathbf{R}^n$.