Isomorphisms and projections *(Sec. 17)*

Definitions: $\underline{A} : \mathcal{V} \to \mathcal{U}$ is *bijective* if it is both injective and surjective (one-to-one and onto). An *isomorphism* is a linear bijection.

If $\mathcal{V}$ and $\mathcal{U}$ are isomorphic, they are abstractly "the same" space. E.g., any finite-dimensional space can be identified with $\mathbf{R}^n$ or $\mathbf{C}^n$ (not uniquely — depends on basis). $\mathcal{L}(\mathcal{V},\mathcal{U})$ is isomorphic to the $m \times n$ matrices, in turn isomorphic to $\mathbf{R}^{mn}$.

Observations

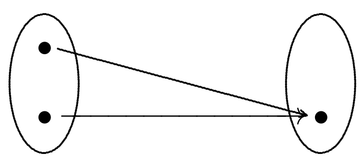
$\underline{A}$ is bijective iff invertible: $\underline{A}^{-1}\underline{A} = \underline{1}$, $\underline{A}\underline{A}^{-1} = \underline{1}$. ($\underline{1} \equiv I =$ identity operator. The two $\underline{1}$'s in the equations refer to different spaces!)

The inverse operator is represented by the inverse matrix. A^{-1} is unique if it exists (on both sides).

One-sided "inverses" are not unique.

Left inverse ($\underline{BA} = \underline{1}$) $\iff$ $\underline{A}$ injective.

Right inverse ($\underline{AB} = \underline{1}$) $\iff$ $\underline{A}$ surjective.



not injective

not surjective

Theorem 17.1. $(\underline{BA})^{-1} = \underline{A}^{-1}\underline{B}^{-1}$ *(if these exist).*

Theorem 17.2. *Two finite-dimensional vector spaces (over the same field) are isomorphic iff they have the same dimension.*

Sketch of proof: Both of them are isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$:

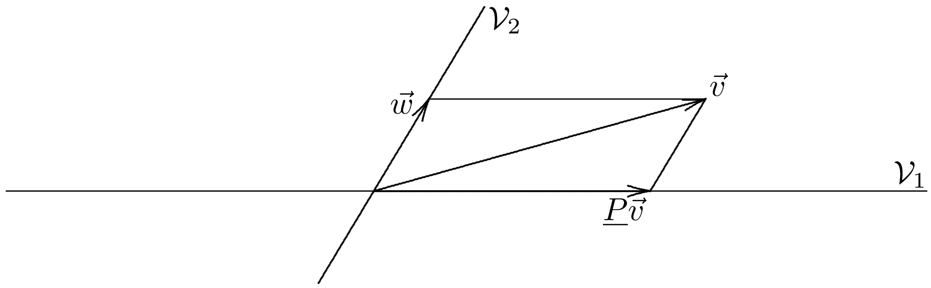
$$\mathcal{V} \xrightarrow{A} \mathbf{R}^n \xleftarrow{B} \mathcal{U}. \qquad \underline{B}^{-1}\underline{A} : \mathcal{V} \to \mathcal{U}.$$

Jargon: An *automorphism* is an isomorphism of $\mathcal{V}$ onto $\mathcal{V}$. The automorphisms make up the *general linear group*, $GL(\mathcal{V})$. $GL(n)$ is the group of invertible $n \times n$ matrices (identifiable with $GL(\mathcal{F}^n)$).

An *endomorphism* is *any* member of $\mathcal{L}(\mathcal{V},\mathcal{V})$ (invertible or not).

Definition: $\underline{P} \in \mathcal{L}(\mathcal{V},\mathcal{V})$ is a *projection* if $\underline{P}^2 = \underline{P}$.

A projection projects *onto* one subspace *along* another (complementary) subspace:



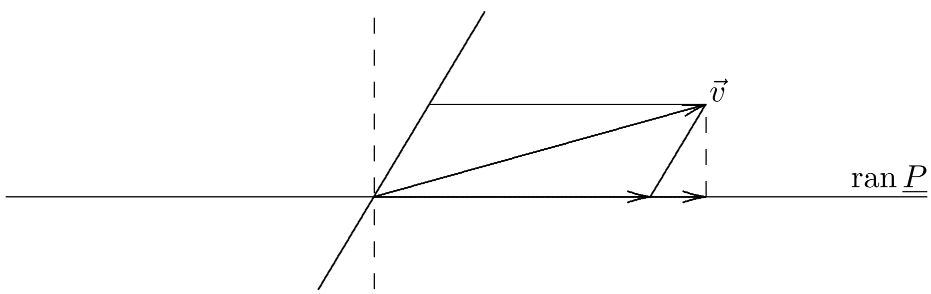
I.e.,

Theorem 17.3. $\mathcal{V} = \text{ran}\ \underline{P} \oplus \ker \underline{P} = \mathcal{V}_1 \oplus \mathcal{V}_2$.

PROOF: Let $\vec{w} = \vec{v} - \underline{P}\vec{v}$. Then $\underline{P}\vec{w} = \vec{0}$, so $\vec{w} \in \ker \underline{P}$. Therefore $\vec{v} = \underline{P}\vec{v} + (\underline{1} - \underline{P})\vec{v} \in \text{ran}\ \underline{P} + \ker \underline{P}$. We must also show that $\text{ran}\ \underline{P} \cap \ker \underline{P} = \{\vec{0}\}$:

$$\vec{u} \in \text{ran} \cap \ker \;\Rightarrow\; \vec{u} = \underline{P}\vec{v} \;\Rightarrow\; \vec{0} = \underline{P}\vec{u} = \underline{P}^2\vec{v} = \underline{P}\vec{v} = \vec{u}.$$

Remark: $\underline{P}$ is defined by the *pair* of subspaces. ran $\underline{P}$ does not determine $\underline{P}$ uniquely:



In an inner product space, the projection onto a given subspace (ran $\underline{P}$) can be made unique by requiring that the complement be *orthogonal*. As a condition on $\underline{P}$, this is $\underline{P} = \underline{P}^*$ (the adjoint of $\underline{P}$, defined in the next section).

In passing we noted that $\underline{1} - \underline{P} =$ projection onto $\ker \underline{P}$ along $\text{ran}\ \underline{P}$. Check this:

$$(\underline{1} - \underline{P})^2 = \underline{1} - \underline{P} - \underline{P} + \underline{P}^2 = \underline{1} - \underline{P}.$$

GENERALIZATION (THEOREM 17.4). $\mathcal{V} = \text{ran}\ \underline{P}_1 \oplus \text{ran}\ \underline{P}_2 \oplus \cdots \oplus \text{ran}\ \underline{P}_R \iff$

$$\underline{P}_k^2 = \underline{P}_k, \qquad \underline{P}_k\underline{P}_l = \underline{0} \text{ if } k \neq l, \qquad \underline{1} = \sum_{k=1}^{R} \underline{P}_k.$$

$\underline{P}_k\vec{v} \equiv$ component of $\vec{v}$ in $\text{ran}\ \underline{P}_k$.