The polar decomposition *(Thm. 27.8)*

Cf. $x + iy = re^{i\theta}, \qquad r > 0, \quad |e^{i\theta}| = 1.$

Definition: $\underline{A}$ is *positive* if $\underline{A}$ is Hermitian and $\vec{v} \cdot \underline{A}\vec{v} \geq 0, \quad \forall \vec{v}$.

(If $\mathcal{F} = \mathbf{C}$, the Hermiticity requirement is redundant — see homework.) This implies that all the eigenvalues of $\underline{A}$ are ≥ 0. Therefore, $\sqrt{\underline{A}}$ is well-defined (and positive):

$$\sqrt{\underline{A}} = \sum_\nu \sqrt{\lambda_\nu}\, \underline{P}_\nu \,.$$

Theorem. *If $\underline{A}$ is* **any** *endomorphism, $\underline{A}\,\underline{A}^*$ and $\underline{A}^*\underline{A}$ are positive.*

Proof: $\vec{v} \cdot \underline{A}^*\underline{A}\vec{v} = \|\underline{A}\vec{v}\|^2 \geq 0$. Similarly for the other case. It is easy to see that the two operators are Hermitian. (Note that they are not equal, in general.)

Consider the positive operator $\sqrt{\underline{A}^*\underline{A}}$. Can we find a unitary operator $\underline{U}$ so that

$$\underline{A} = \underline{U}\sqrt{\underline{A}^*\underline{A}} \quad ?$$

[Warning: Bowen & Wang's $\underline{U}$ is my $\sqrt{\underline{A}^*\underline{A}}\,$!] Assume for simplicity that $\underline{A}$ is invertible. Equivalently (see homework), $\sqrt{\underline{A}^*\underline{A}}$ is invertible (i.e., $\sqrt{\underline{A}^*\underline{A}}$ is positive *definite*; $0 \notin \sigma\big(\sqrt{\underline{A}^*\underline{A}}\big)$). Then $\underline{U}$ must be

$$\underline{U} = \underline{A}(\underline{A}^*\underline{A})^{-\frac{1}{2}}.$$

Let's check unitarity:

$$\underline{U}^*\underline{U} = (\underline{A}^*\underline{A})^{-\frac{1}{2}}\underline{A}^*\underline{A}(\underline{A}^*\underline{A})^{-\frac{1}{2}} = (\underline{A}^*\underline{A})^0 = \underline{1}\,;$$
$$\underline{U}\,\underline{U}^* = \underline{A}(\underline{A}^*\underline{A})^{-1}\underline{A}^* = \underline{A}\,\underline{A}^{-1}\underline{A}^{*-1}\underline{A}^* = \underline{1}.$$

(Actually, one of these would have sufficed; our proof is limited to finite-dimensional endomorphisms anyway, since only for them do we have a spectral theorem so far.)

One can prove (see book) that $\sqrt{\underline{A}^*\underline{A}}$ is the *only* positive Hermitian operator $\underline{B}$ with the property that $\underline{U} \equiv \underline{A}\,\underline{B}^{-1}$ is unitary.

Similarly, $\underline{A} = \sqrt{\underline{A}\,\underline{A}^*}\,\underline{U}$ (with the *same* $\underline{U}$). To see that the $\underline{U}$'s are the same, note that this second type of polar decomposition is also unique and then write

$$\underline{A} = \underline{U}\sqrt{\underline{A}^*\underline{A}} = \underline{U}\sqrt{\underline{A}^*\underline{A}}\,\underline{U}^{-1}\underline{U}\,;$$

since $\underline{U}\sqrt{\underline{A}^*\underline{A}}\,\underline{U}^{-1}$ is positive, it must equal $\sqrt{\underline{A}\,\underline{A}^*}$.

If $\underline{A}$ is not invertible, things become more complicated and $\underline{U}$ is not unique. (It can map $\ker \underline{A}$ onto $(\operatorname{ran} \underline{A})^\perp$ in an arbitrary isometric way.) Let's see whether you can handle this case as a homework problem.

Putting all this together, we have proved a **theorem**, which it should not be necessary to restate.

Sylvester's formula *(cf. (28.18))*

If $\underline{A}$ has diagonal Jordan form ($\underline{A} = \sum \lambda_\nu \underline{P}_\nu$), then

$$\underline{P}_\nu = \frac{\displaystyle\prod_{\mu\neq\nu} (\lambda_\mu \underline{1} - \underline{A})}{\displaystyle\prod_{\mu\neq\nu} (\lambda_\mu - \lambda_\nu)}$$

(where μ varies from 1 to L in each product).

Proof: $\underline{P}_\nu = f(\underline{A})$ where f is any function satisfying

$$f(\lambda_\nu) = 1, \qquad f(\lambda_\mu) = 0 \quad \text{if } \mu \neq \nu.$$

Such a function is

$$f(x) = \frac{\prod_{\mu\neq\nu} (\lambda_\mu - x)}{\prod_{\mu\neq\nu} (\lambda_\mu - \lambda_\nu)} \,.$$

Proof of the "Feynman diagram" algorithm for the invariants μ_j in terms of traces of powers of $\underline{A}$

We want to expand $\det(\underline{A} - \lambda)$ as a polynomial in λ. It's convenient to write

$$\det(\underline{A} - \lambda) = (-\lambda)^N \det(\underline{1} - \lambda^{-1}\underline{A}).$$

Let $z \equiv \lambda^{-1}$. If z is sufficiently small, $\ln(\underline{1} - z\underline{A})$ will be well-defined. [Recall that

$$\ln(\underline{1} - \underline{A}) = -\sum_{n=1}^{\infty} \frac{1}{n} \underline{A}^n$$

makes sense for $\underline{A}$ such that the series converges. Alternatively,

$$\ln(\underline{1} - z\underline{A}) = \sum_\nu \ln(\underline{1} - z\lambda_\nu)\underline{P}_\nu + \text{nilpotent terms} \qquad (\#)$$

is well-defined if, for all ν, $\underline{1} - z\lambda_\nu$ is not on the nonpositive real axis.]

Lemma. $\det \underline{B} = e^{\operatorname{tr}\ln \underline{B}}$ *if* $\ln \underline{B}$ *is defined.*

This is an important theorem in its own right.

PROOF:

Case 1: $\underline{B}$ diagonalizable. Look at the matrices in a basis where B is diagonal.

$$B = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}; \qquad \ln B = \begin{pmatrix} \ln \lambda_1 & & 0 \\ & \ln \lambda_2 & \\ 0 & & \ddots \end{pmatrix};$$

so

$$\text{tr } \ln \underline{B} = \sum_{j=1}^{N} \ln \lambda_j \,;$$

$$e^{\text{tr } \ln \underline{B}} = \prod_{j=1}^{N} e^{\ln \lambda_j} = \prod_{j=1}^{N} \lambda_j = \det \underline{B}.$$

This is an operator (basis-independent) result, since det and tr are invariant under similarity transformations, and ln and exp are *covariant* thereunder [i.e., $S \ln B\, S^{-1} = \ln (SBS^{-1})$, etc.].

Case 2: $\underline{B}$ has nondiagonal Jordan canonical form.

ARGUMENT 1: Such $\underline{B}$'s form a lower-dimensional hypersurface in the space of all endomorphisms. That is, any matrix with nondiagonal JCF is a limit of a sequence of matrices with diagonal JCF; indeed, any matrix with coincident eigenvalues is a limit of matrices with distinct eigenvalues, obtained by slightly perturbing one or more elements of the matrix. [This is true even though the corresponding diagonal matrices *do not* converge to the nondiagonal JCF. A sequence of 0's can't converge to a 1!] Since all the functions involved are continuous functions of $\underline{B}$, the result follows from Case 1.

ARGUMENT 2: We noted earlier that B in Jordan form, with diagonal elements λ_j, implies that $f(B)$ is upper triangular, with diagonal elements $f(\lambda_j)$. Therefore, $\operatorname{tr} \ln B = \sum_j \ln \lambda_j$, $\det B = \prod_j \lambda_j$, and hence the rest of the proof given for Case 1 applies after all!

Now we resume the derivation of the Feynman-diagram algorithm. Expand $\det(\underline{1} - z\underline{A}) = \exp[\text{tr } \ln(\underline{1} - z\underline{A})]$ as a power series in z. If $\underline{A}$ is $N \times N$, this will in fact terminate with the z^N term and give us $\det(\underline{1} - z\underline{A})$. (Our proof gives us this for small z, but the result is then correct for all z since it's a purely algebraic fact. We know that the characteristic invariants are polynomials in the matrix elements of A; the only question is what the coefficients are. Alternatively, one can appeal to the uniqueness of analytic continuations of analytic functions.)

We expand first ln and then exp, getting

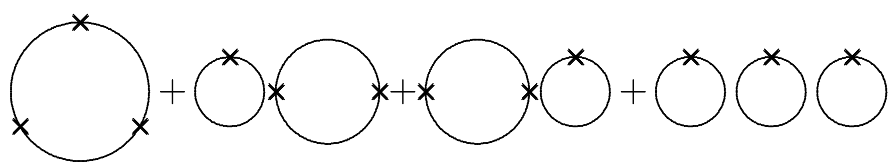
$$\exp\left\{-\mathrm{tr}\sum_{n=1}^{\infty}\frac{z^n A^n}{n}\right\} = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left\{\mathrm{tr}\sum_{n=1}^{\infty}\frac{z^n A^n}{n}\right\}^p$$
$$\equiv \sum_{j=0}^{\infty} z^j(-1)^j \mu_j\,.$$

Therefore, each term in μ_j has the form

$$\frac{(-1)^{j+p}}{p!}\prod_{i=1}^{p}\frac{\mathrm{tr}\,(A^{n_i})}{n_i}$$

for some p, $n_1, \ldots, n_p$ such that $\sum_{i=1}^{p} n_i = j$.

Let us represent each term by an "ordered" ring diagram; for example,

$\mu_3 =$ 

$\mathrm{tr}\,(A^3)$ $\quad (\mathrm{tr}\,A)\mathrm{tr}\,(A^2)$ $\quad \mathrm{tr}\,(A^2)(\mathrm{tr}\,A)$ $\quad (\mathrm{tr}\,A)^3$

We note that the two middle terms can be combined. In general, terms which differ only by the ordering of rings with different numbers of vertices can be combined. The number of ordered diagrams equivalent to a given unordered diagram is the number ($p!$) of distinct permutations of rings divided by the numbers of permutations that just interchange rings with the same number of vertices:

$$\frac{p!}{m_1!\, m_2! \,\cdots\, m_{\max n_i}!}$$

where m_k is the number of rings containing k vertices. (The reason that permutations of the latter type don't count is that they do not yield distinct ordered diagrams.) Therefore, the coefficient associated with each *unordered* ring diagram is

$$\frac{(-1)^{j+p}}{\left(\prod_{i=1}^{p} n_i\right)\left(\prod_{k=1}^{\max n_i} m_k!\right)}\,.$$

This multiplies $\prod_{i=1}^{p}\mathrm{tr}\,(A^{n_i})$, symbolized by the rings in the corresponding diagram.

This is the rule stated in an earlier lecture, QED.