Math. 640 13 December 1994

Final Examination – In-class part – Solutions

(150 points of 200)

1. *(25 pts.)* Let $\mathcal{V}$ be an inner-product space and $\mathcal{U}$ a subspace of $\mathcal{V}$, with $\dim \mathcal{V} = n$ and $\dim \mathcal{U} = p$. What are the dimensions of the following spaces?

(a) $\mathcal{V}/\mathcal{U}$ (b) $\mathcal{U}^{\perp}$ (c) $\mathcal{V}+\mathcal{U}$ (d) $\mathcal{V}\cap\mathcal{U}$ (e) $\mathcal{V}\otimes\mathcal{U}$

(a) $n-p$ (b) $n-p$ (c) n $(\mathcal{V}+\mathcal{U}=\mathcal{V})$ (d) p $(\mathcal{V}\cap\mathcal{U}=\mathcal{U})$ (e) np

2. *(40 pts.)* Let $M = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$.

(a) Find e^{tM} and use it to solve the differential equation problem

$$\frac{d\vec{x}}{dt} = M\vec{x}(t), \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Method 1: Power series. $M^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow M^{2k} = 4^k I, \quad M^{2k+1} = 4^k M$. So

$$\begin{aligned} e^{tM} &= \sum_{n=0}^{\infty} \frac{(tM)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{4^k t^{2k}}{(2k)!}\, I + \sum_{k=0}^{\infty} \frac{4^k t^{2k+1}}{(2k+1)!}\, M \\ &= \cosh(2t) I + \tfrac{1}{2}\sinh(2t) M \\ &= \begin{pmatrix} \cosh 2t & 2\sinh 2t \\ \frac{1}{2}\sinh 2t & \cosh 2t \end{pmatrix}. \end{aligned}$$

Now $\vec{x}(t) = e^{tM}\vec{x}(0) = \begin{pmatrix} 2\cosh 2t + 2\sinh 2t \\ \sinh 2t + \cosh 2t \end{pmatrix}$. *Check:* $\left.\frac{d}{dt} e^{tM}\right|_{t=0} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$.

Method 2: Spectral decomposition.

$$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 \;\Rightarrow\; \lambda = \pm 2.$$

Eigenvector for $\lambda = 2$: $x - 2y = 0 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Eigenvector for $\lambda = -2$: $x + 2y = 0 \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Therefore, a diagonalizing matrix is $S = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$. Then $S^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$, and

$$M = SDS^{-1}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then

$$\begin{aligned} e^{tM} &= Se^{tD}S^{-1} \\ &= \frac{1}{4}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh 2t & 2\sinh 2t \\ \frac{1}{2}\sinh 2t & \cosh 2t \end{pmatrix}. \end{aligned}$$

as before.

(b) Find a polar decomposition of M (either of the two possibilities).

$M^* = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \Rightarrow M^*M = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \Rightarrow \sqrt{M^*M} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow$

$$U = M(M^*M)^{-1/2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} M &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \\ &= (\text{unitary}) \times (\text{Hermitian}). \end{aligned}$$

Alternatively, $M = \sqrt{MM^*}\,U = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (But note that

$$M = V\sqrt{MM^*} \quad \text{or} \quad M = \sqrt{M^*M}\,V$$

won't work, because

$$V = (M^*M)^{-1/2}M = \begin{pmatrix} 0 & 4 \\ \frac{1}{4} & 0 \end{pmatrix}$$

is not unitary.)

3. *(15 pts.)* Find the kernel of this operator, and say as much about its range as you can: $\underset{\sim}{A}\colon \mathcal{V} \to \mathcal{H}$, where $\mathcal{V}$ is the space of twice-differentiable functions on $(0,\pi)$ satisfying $f(0) = 0 = f(\pi)$, $\mathcal{H}$ is the space of square-integrable functions on $(0,\pi)$, and

$$\underset{\sim}{A}f = \frac{d^2 f}{dt^2} + 4f.$$

The kernel is the space of solutions of

$$f'' + 4f = 0, \quad f(0) = 0 = f(\pi),$$

which consists of the scalar multiples of $f_0(t) = \sin 2t$.

The range is the set of square-integrable $g(t)$ such that

$$f'' + 4f = g, \quad f(0) = 0 = f(\pi)$$

has a solution $f \in \mathcal{V}$. Integration by parts shows that such a g must be orthogonal to the kernel. (In fact, the range is only "slightly smaller" than the orthogonal complement of the kernel, and would be equal to the orthogonal complement if we made the domain $\mathcal{V}$ slightly larger, to accommodate "rough" solutions f. These functions have Fourier series

$$g(t) = \sum_{n=1}^{\infty} b_n \sin(nt), \qquad b_2 = 0, \quad \sum_{n=1}^{\infty} |b_n|^2 < \infty.$$

Of course, you weren't expected to say all this.) You can also construct f by variation of parameters and find that the construction fails unless g is orthogonal to f_0.

4. *(20 pts.)*

(a) Let N be a fixed integer (≥ 2) and $\{\vec{u}_j\}_{j=1}^{N}$ be a basis for a vector space. Denote by x^j the coordinates of an arbitrary vector with respect to this basis: $\vec{v} = \sum_{j=1}^{N} x^j \vec{u}_j$. Introduce a new basis by

$$\begin{aligned} \vec{e}_1 &= \vec{u}_1 , \\ \vec{e}_j &= \vec{u}_j - \vec{u}_{j-1} \quad \text{for } j = 2, \ldots, n. \end{aligned}$$

and denote the new coordinates by y^j: $\vec{v} = \sum_{j=1}^{N} y^j \vec{e}_j$. Find the formulas expressing the xs in terms of the ys and vice versa.

The matrix in the formulas

$$\text{(old coordinates)} \leftarrow \text{(new coordinates)}$$

is the transpose of that in the formulas

$$\text{(new basis)} \leftarrow \text{(old basis)}.$$

Thus

$$\begin{aligned} x^1 &= y^1 - y^2 , \ldots \\ x^j &= y^j - y^{j+1} , \ldots \\ x^N &= y^N . \end{aligned}$$

We can solve from the bottom up to get

$$\begin{aligned} y^N &= x^N , \\ y^{N-1} &= x^{N-1} + x^N , \ldots \\ y^j &= \sum_{k=j}^{N} x^k , \ldots . \end{aligned}$$

Three-dimensional example:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \overset{\text{transpose}}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \overset{\text{inverse}}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Consider a hyperplane ($(N-1)$-dimensional subspace) defined by an equation of the form

$$\sum_{k=i}^{j} x^k = 0$$

for some choice of i and j satisfying $1 \le i \le j \le N$. (Remember, the k is an index, **not** an exponent!) Find the expression of such a hyperplane in terms of the y coordinates, and summarize in words (distinguishing different cases as necessary).

(This is a "real problem" that arose two weeks ago in the study of numerical integration over a tetrahedron or higher-dimensional simplex.)

For convenience, introduce the convention $y^{n+1} \equiv 0$. Then

$$\sum_{k=i}^{j} x^k = \sum_{k=i}^{j} \left(y^k - y^{k+1}\right) = y^i - y^{j+1}.$$

Thus

(1) if $j = N$, the equation is $y^i = 0$ and the hyperplane is one of the coordinate hyperplanes;

(2) if $j \ne N$, let $l = j+1$; then the equation is $y^i = y^l$ and the hyperplane is one that slices through the origin at a 45° angle in the (y^i, y^l) plane, but contains the other $N-2$ coordinate axes.

5. *(25 pts.)* Find the Jordan canonical form of $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$,

and a corresponding Jordan basis.

(This problem was recently discussed over the Internet as one that *Mathematica* can't do.)

The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & -1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda \left[-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -\lambda \\ -1 & 1 \end{vmatrix} \right]$$

$$= \lambda^2(\lambda^2 - 1) + \lambda^2 = \lambda^4.$$

Thus $\lambda = 0$ is the only eigenvalue.

The eigenvectors, therefore, are just the kernel of A. The matrix row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

With the notation

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

for vectors, we therefore have

$$w = -z, \qquad x = y.$$

Thus two independent eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Associated vectors must satisfy $A\vec{u} = \vec{v}$ for some $\vec{v}$ in the span of $\vec{v}_1$ and $\vec{v}_2$. But $\vec{v}$ must also be in the range of A. Recalling that the range is the span of the columns, and looking back at the original matrix, we see that $\vec{v}_2$ is in the range but any linear combination containing a $\vec{v}_1$ component is not. So we have to solve $A\vec{u} = \vec{v}_2$. The augmented matrix of this system row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the equations are

$$w = 1 - z, \qquad x = y.$$

At this point we must be very careful: We will need to find another associated vector solving $A\vec{w} = \vec{u}$. Therefore, we must choose the parameters z and y so that $\vec{u}$ is in the range of A.

$$\begin{pmatrix} 1-z \\ y \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The simplest choice is $y = 0 = \alpha$, $z = 1 = \beta$. Thus $\vec{u} = (0, 0, 0, 1)$ (transposed), and the augmented matrix of the equation for $\vec{w}$ row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

That is, $w = -z$, $x = y - 1$. I choose $z = 0$, $y = 1$; $\vec{w} = (0, 0, 1, 0)$.

Thus the Jordan form is

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and it represents A with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{u}, \vec{w}\} =$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Check: See the attached *MathCAD* worksheet.

6. *(Essay – 25 pts.)* Tell what you know about **ONE** of these:
   (A) The Fredholm alternative.
   (B) Tensors (various definitions).
   (C) Factor spaces.
   (D) Relationships among antisymmetric operators, rotations, vector cross product, and curl.