Math. 640 30 September 1994

Test A – Solutions

1. *(30 pts.)* Let $\underset{\sim}{A}\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear function with matrix $\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 4 & 2 & 6 \end{pmatrix}$. Find bases for

(a) ker $\underset{\sim}{A}$

The kernel is the solution space of the homogeneous equation $\underset{\sim}{A}\vec{x} = 0$. So, row-reduce the matrix to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

getting the equivalent homogeneous equations

$$\begin{aligned} a + \quad c &= 0, \\ b + c &= 0. \end{aligned}$$

Choosing c arbitrarily, one can solve uniquely for a and b. The simplest choice is $c = -1$, yielding the single basis vector

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(b) ran $\underset{\sim}{A}$

The range is the span of the columns, so we rewrite the columns as rows and row-reduce:

$$\begin{pmatrix} 0 & 1 & 4 \\ 2 & 0 & 2 \\ 2 & 1 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus a basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}.$$

(This is not the only correct answer, of course.)

(c) ker $\underset{\sim}{A}^*$

Method 1: Transpose the matrix and proceed as in (a). The row reduction is identical to that in (b), and you get the single basis vector

$$\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}.$$

Method 2: The kernel of $\underset{\sim}{A}^*$ is the orthogonal complement of the range of $\underset{\sim}{A}$, so we should be able to get the answer from the solution to (b).

Method 2a: Take the vector cross product of the two vectors found in (b). (This works only in dimension 3!)

Method 2b: The condition that a vector is orthogonal to both basis vectors in (b) is

$$\begin{aligned} a + \qquad c &= 0, \\ b + 4c &= 0. \end{aligned}$$

But this is the same system you have to solve in Method 1.

2. *(15 pts.)* Let $\mathcal{V}$ and $\mathcal{U}$ be subspaces of a vector space $\mathcal{X}$.

(a) Explain what is meant by $\mathcal{V}+\mathcal{U}$.

$$\mathcal{V}+\mathcal{U} = \{\vec{v}+\vec{u}\colon \vec{v}\in\mathcal{V},\ \vec{u}\in\mathcal{U}\}.$$

(b) State and prove the formula for the dimension of $\mathcal{V}+\mathcal{U}$.

$$\dim(\mathcal{V}+\mathcal{U}) = \dim\mathcal{V} + \dim\mathcal{U} - \dim(\mathcal{V}\cap\mathcal{U}).$$

Proof: Let $\{\vec{w}_1,\ldots,\vec{w}_r\}$ be a basis for $\mathcal{V}\cap\mathcal{U}$. Extend this set to a basis for $\mathcal{V}$ (thereby adding $\dim\mathcal{V}-r$ vectors that are in $\mathcal{V}$ but not in $\mathcal{U}$), and similarly extend it to a basis for $\mathcal{U}$. Then combine these two sets to form a set $\mathcal{B}$. It is clear that the number of vectors in $\mathcal{B}$ is the number on the right side of the equation to be proved. The proof will be complete once we verify that $\mathcal{B}$ is a basis for $\mathcal{V}+\mathcal{U}$. Well, every vector in $\mathcal{V}+\mathcal{U}$ is a sum of two vectors, each of which can be expanded in terms of one of the two bases we constructed at the intermediate stage, so $\mathcal{B}$ spans. To show that it is linearly independent, suppose that some linear combination of its elements adds to zero. Then we have three vectors

$$\vec{w}+\vec{v}+\vec{u} = 0$$

where $\vec{w}\in\mathcal{V}\cap\mathcal{U}$, $\vec{v}\notin\mathcal{U}$ (if $\vec{v}\neq 0$), and $\vec{u}\notin\mathcal{V}$ (if $\vec{u}\neq 0$). Since $\vec{v}\in\mathcal{V}$, we have $\vec{w}+\vec{v}\in\mathcal{V}$ equal to $-\vec{u}\notin\mathcal{V}$, which is a contradiction unless all the vectors are zero.

3. *(28 pts.)* Let $\mathcal{P}_3$ be the space of cubic polynomials, and consider the differential operator

$$L = \frac{d^2}{dt^2} + t\,\frac{d}{dt}$$

regarded as a linear function from $\mathcal{P}_3$ to $\mathcal{P}_3$. (Thus $Lp(t) \equiv p''(t) + tp'(t)$.)

(a) Find the matrix representing L with respect to the usual basis of $\mathcal{P}_3$ (the power functions $\{t^3, t^2, t, 1\}$).

$$\begin{aligned} L(t^3) &= 3t^3 + 6t, \\ L(t^2) &= 2t^2 + 2, \\ L(t) &= t, \\ L(1) &= 0. \end{aligned}$$

So by the "kth column rule" the matrix is

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

(b) What is the kernel of L?

The matrix quickly row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This betokens that if $L(at^3 + bt^2 + ct + d) = 0$, then $a = b = c = 0$ but d is arbitrary. So the kernel consists of the constant functions.

(c) Rephrase question (b) so that a Math. 308* student would understand it.

"Find all cubic polynomials that solve the differential equation $y'' + ty' = 0$." (Incidentally, the other (linearly independent) solution is $\int e^{-t^2/2}\,dt$.)

(d) What is the dimension of the range of L?

Dimension of domain − dimension of kernel $= 4 - 1 = 3$. Indeed, we can see directly that the three nonzero columns of the matrix are independent.

4. *(15 pts.)* Let $\mathcal{S}$ be an arbitrary subset of an inner product space $\mathcal{V}$.

(a) Prove that $\mathcal{S} \subseteq \mathcal{S}^{\perp\perp}$.

If $\vec{v} \in S$, then by definition of $\mathcal{S}^{\perp}$, we have $\vec{v} \cdot \vec{u} = 0$ for all $\vec{u} \in \mathcal{S}^{\perp}$. But then by definition of $\mathcal{S}^{\perp\perp}$ (and the essential symmetry of the inner product), $\vec{v} \in \mathcal{S}^{\perp\perp}$.

(b) What additional information is needed, under various circumstances, to ensure that $\mathcal{S} = \mathcal{S}^{\perp\perp}$?

First, it is necessary that $\mathcal{S}$ be a sub*space*, not just an arbitrary sub*set*. If $\mathcal{S}$ is infinite-dimensional, $\mathcal{S}$ also needs to be (topologically) *closed*. (A finite-dimensional subspace is automatically closed, so this is not a separate condition in that case.)

5. *(12 pts.)* If $\underline{A}$ is a nonsingular linear transformation, are these statements true or false? Explain.

(a) $\underline{A}^{-1}$ is a linear transformation.

TRUE: This is implicit in all our calculations of matrix inverses. For a formal proof, observe

$$\begin{aligned} \underline{A}^{-1}(r\vec{x} + \vec{y}) &= \underline{A}^{-1}(r\underline{A}\vec{u} + \underline{A}\vec{v}) \quad \text{for some } \vec{u} \text{ and } \vec{v} \\ &= \underline{A}^{-1}\underline{A}(r\vec{u} + \vec{v}) \\ &= r\vec{u} + \vec{v} \equiv r\underline{A}^{-1}\vec{x} + \underline{A}^{-1}\vec{y}. \end{aligned}$$

(b) The mapping $\underline{A} \to \underline{A}^{-1}$ is a linear transformation.

FALSE: $(\underline{A} + \underline{B})^{-1} \neq \underline{A}^{-1} + \underline{B}^{-1}$; $\quad (r\underline{A})^{-1} = \frac{1}{r}\,\underline{A}^{-1} \neq r\underline{A}^{-1}$.

---

* introductory differential equations, with no linear algebra prerequisite