Math. 640 7 November 1994

Test B – Solutions

1. *(15 pts.)* Without extensive calculations, tell whether each of these matrices – can be diagonalized by a unitary matrix; can be diagonalized, but only by a nonunitary matrix; or can't be diagonalized at all.

(a) $\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 2+i \\ 2-i & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$

(a) This is clearly going to have a nondiagonal Jordan form. Let's look at it more carefully: The characteristic equation is $(2-\lambda)^2 = 0$, which has a double root. But if it had two linearly independent eigenvectors with that eigenvalue, it would be a multiple of the identity matrix. Therefore, this matrix cannot be diagonalized.

(b) This matrix is Hermitian, so it can be diagonalized by a unitary.

(c) This has characteristic equation $\lambda^2 - 4\lambda + 2 = 0$, which has distinct real roots ($\lambda = 2 \pm \sqrt{2}$). Therefore, it can be diagonalized. Since the roots are real, if the diagonalizing matrix were unitary (eigenvectors orthogonal), the original matrix would have to be Hermitian, which it is not. Therefore, this matrix cannot be diagonalized by a unitary.

Remark: If the distinct roots were complex, it would be possible for the diagonalizing matrix to be unitary although the original matrix is not Hermitian. (I.e., the eigenvectors are orthogonal but the eigenvalues are not all real. Such a matrix is *normal*.) An example is $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, with eigenvalues $\lambda = 2 \pm i$. It is easy to check that this matrix commutes with its adjoint, which is another characterization of a normal matrix. Five points extra credit were awarded for understanding this extra subtlety.

2. *(25 pts.)* Let $\mathcal{U}$ be a subspace of $\mathcal{V}$, and let $\mathcal{W}$ be a direct complement of $\mathcal{U}$ (not necessarily an *orthogonal* direct complement). Thus $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$. Finally, let $\mathcal{X}$ be the factor space $\mathcal{V}/\mathcal{U}$.

   (a) Prove that every element of $\mathcal{X}$ contains exactly one element of $\mathcal{W}$.

Existence: Given $\overline{x} \in \mathcal{X}$, consider an arbitrary $\vec{x} \in \overline{x}$. By definition of direct complement, $\vec{x}$ has a decomposition of the form $\vec{u} + \vec{w}$. By definition of factor space, $\vec{w} \in \overline{x}$.

Uniqueness: If there are two such vectors $\vec{w}$ in $\overline{x}$, then their difference must be in $\mathcal{U}$ — as well as in $\mathcal{W}$. Since the sum of subspaces is direct, this difference vector must be zero.

   (b) Prove that the one-to-one correspondence between $\mathcal{X}$ and $\mathcal{W}$ set up by (a) is a linear isomorphism.

As the wording of the question implies, surjectivity (onto $\mathcal{W}$) is obvious, because every vector $\vec{w}$ belongs to some coset. What needs to be checked is linearity. That follows from the definition of the algebraic operations in $\mathcal{X}$: Given $\overline{x}_j = \vec{w}_j + \mathcal{U} \in \mathcal{X}$, we can form linear combinations and lump the $\mathcal{U}$ contributions together:

$$\sum \alpha^j \overline{x}_j = \left(\sum \alpha^j \vec{w}_j\right) + \mathcal{U}.$$

This shows that the $\vec{w}$s depend linearly on the $\overline{x}$s.

   (c) Why do you think that the notion of factor space might be especially useful when the basic vector space ($\mathcal{V}$, here) is *not* an inner product space? (In other words, why is it easy to avoid working with factor spaces when $\mathcal{V}$ does have an inner product?)

Factor spaces and orthogonal complements are unique, but more general direct complements are not. The elements of an orthogonal complement provide an unambiguous way of labeling the elements of $\mathcal{V}/\mathcal{U}$. For many purposes, therefore, we can think about the (relatively concrete) orthogonal complement and have no need to introduce the more abstract factor space. Doing the same thing with a nonorthogonal direct complement is arbitrary and inelegant.

3. *(15 pts.)* Let $\underset{\sim}{A}\colon \mathcal{V} \to \mathcal{V}$ be a finite-dimensional linear automorphism. Let $\mathcal{V}_\mu$ be the subspace of $\mathcal{V}$ consisting of all eigenvectors of $\underset{\sim}{A}$ with eigenvalue μ.

   (a) Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis for $\mathcal{V}_\mu$, and suppose that it has been extended to a basis for $\mathcal{V}$. Indicate schematically the form of the matrix representing $\underset{\sim}{A}$ with respect to this basis.

$$\begin{pmatrix} \begin{matrix} \mu & & 0 \\ & \mu & \\ 0 & & \ddots \end{matrix} & \mathrm{B} \\ 0 & \mathrm{C} \end{pmatrix},$$

where the diagonal block is $k \times k$, and the blocks $\mathbf{B}$ and $\mathbf{C}$ are not zero, in general.

   (b) Use the result of (a) to prove that the geometric multiplicity of μ is less than or equal to its algebraic multiplicity.

$\det(\underset{\sim}{A} - \lambda) = (\mu - \lambda)^k \det(\mathbf{C} - \lambda)$ has μ as a root of multiplicity at least k. (It could be greater than k, since μ might be a root of $\det(\mathbf{C} - \lambda)$.) The multiplicity of the root is, by definition, the algebraic multiplicity, while k is the geometric multiplicity.

4. *(15 pts.)* Consider the partial differential equation $\nabla^2\phi(\mathbf{x}) = \rho(\mathbf{x})$ for $\mathbf{x} \in \Omega \subset \mathbf{R}^3$ with the boundary condition that $\phi(\mathbf{x}) = 0$ for $\mathbf{x}$ on the boundary of Ω. (Here ρ is a known function and ϕ is the unknown to be found.) A standard way of solving this problem is this:

   (1) One observes that
$$\tilde{\phi}(\mathbf{x}) \equiv \int_{\mathbf{R}^3} \frac{-\rho(\mathbf{y})\, d\mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|}$$
   solves $\nabla^2\tilde{\phi}(\mathbf{x}) = \rho(\mathbf{x})$ throughout all of $\mathbf{R}^3$.

   (2) One observes that $\psi \equiv \phi - \tilde{\phi}$ must satisfy $\nabla^2\psi(\mathbf{x}) = 0$ in Ω with inhomogeneous boundary data that can be calculated from $\tilde{\phi}$.

   (3) One knows from a treatise on PDEs that the only solution of $\nabla^2\psi(\mathbf{x}) = 0$ in Ω with *homogeneous* boundary data is $\psi = 0$.

   (4) One finds ψ by some standard analytical or numerical method.

   Within this scenario, find an instance of each of the following:

   (a) a linear operator whose domain is an infinite-dimensional vector space

The Laplacian operator ∇^2. The domain can be taken to be either all twice-differentiable functions on Ω, or all such functions that vanish on the boundary.

(b) a kernel

There is more than one correct answer. The kernel of the second operator in (a) contains just the zero vector, as stated in (3). The kernel of the first operator in (a) is more interesting, since it enables us to answer (c) in a nontrivial way. This latter kernel consists of all solutions of Laplace's equation on Ω (without regard to boundary conditions). (Students heading for a legal career might also argue that the Coulomb potential, $\frac{-1}{4\pi|\mathbf{x}-\mathbf{y}|}$, is an "integral kernel" and thus satisfies all the requirements of the question as stated.)

(c) a coset or a factor space

The quotient of the domain (of the first operator in (a)) by its kernel is the space of equivalence classes of solutions of the inhomogeneous equations $\nabla^2\phi = \rho$, two solutions being equivalent if they solve the equation for the same ρ. Compare (2).

(d) the Fredholm principle, "Uniqueness implies existence". (I.e., the range of the operator in (a) is its entire codomain.)

The Laplacian operator with Dirichlet boundary conditions is self-adjoint, so (3) implies that its range is the orthogonal complement of the zero vector — the whole space. (Analytically this is a bit sloppy, but the question speaks of "Fredholm principle", not "Fredholm theorem".)

5. *(30 pts.)* Find an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors of

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

Hint: One of the eigenvalues is 7.

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 3-\lambda \\ 2 & 2 \end{vmatrix}$$

$$= (3-\lambda)^3 - 4(3-\lambda) - 4(3-\lambda) + 8 + 8 - 4(3-\lambda) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27 - 36 + 12\lambda + 16$$

$$= -\lambda^3 + 9\lambda^2 - 15\lambda + 7 = 0.$$

Long division by $\lambda - 7$ leaves a residual factor $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. So 1 is a double root.

Eigenvector for $\lambda = 7$:

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \quad \text{row-reduces to} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

with normalized solution

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Eigenvectors for $\lambda = 1$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \quad \text{row-reduces to} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two independent solutions are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

These need to be orthonormalized by the Gram–Schmidt construction:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

It follows that an orthonormal basis for the whole space is

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$