Math. 640 October 2018

Midterm Test – Solutions

1. *(15 pts.)* Consider the vector space $\mathcal{C}$ of all real-valued continuous functions defined on $[0,1]$. For each of the following conditions, decide whether it defines a subspace of $\mathcal{C}$.

   (a) $f(0) = 0 = f(1)$.

Yes.

   (b) The graph of $f(x)$ is symmetric with respect to reflection about the midpoint, $x = \frac{1}{2}$.

Yes.

   (c) $f(0) + f(1) = 1$.

No (the condition is not *homogeneous* linear).

2. *(20 pts.)* Let $\mathcal{V}$ be the space of polynomials of degree less than or equal to 2 and $\underset{\sim}{A} : \mathcal{V} \to \mathcal{V}$ be the operator $\dfrac{d}{dt} + 5$. (That is, $[\underset{\sim}{A}\, p](t) = p'(t) + 5p(t)$.)

   (a) Find the matrix representing $\underset{\sim}{A}$ with respect to the usual basis of $\mathcal{V}$ (the power functions $\{t^2, t, 1\}$).

The quickest method is to use the rule that the j th column is the image of the j th basis vector.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix}.$$

Or, you can read off the matrix from the formula

$$\left(\frac{d}{dt} + 5\right)(at^2 + bt + c) = (2at + b) + (5at^2 + 5bt + 5c) = 5at^2 + (2a + 5b)t + (b + 5c).$$

   (b) What is the kernel of $\underset{\sim}{A}$?

The matrix row-reduces to the identity, so the kernel contains only the zero polynomial.

   (c) What is the range of $\underset{\sim}{A}$?

From (b) and the rank-nullity theorem, the range is all of $\mathcal{V}$. (Or, row-reduce the transpose to the identity matrix.)

   (d) What is the rank of $\underset{\sim}{A}$?

3

(This problem was intended to have nontrivial kernel and range, but I modified it from an old test and forgot to retain that property. Some of you got the wrong matrix in (a) and thus got a more interesting problem nevertheless!)

3. *(15 pts.)* Use the Gram–Schmidt construction to get an orthonormal basis for

$$\text{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}.$$

Let's call the given vectors $\vec{v}_j$ and the orthonormalized vectore $\vec{u}_j$. At the first step, $\vec{v}_1\cdot\vec{v}_2=0$ already, so $\vec{u}_1$ and $\vec{u}_2$ are just normalized versions of their $\vec{v}$ s. Note that $\|\vec{v}_1\|^2=\|\vec{v}_2\|^2=4$. At the next step,

$$\vec{v}_3\cdot\vec{v}_1=1+2+3+4=10, \qquad \vec{v}_3\cdot\vec{v}_2=1-2+3-4=-2.$$

So the projection onto the 1–2 plane is

$$\frac{10}{4}\vec{v}_1-\frac{2}{4}\vec{v}_2=\begin{pmatrix}2\\3\\2\\3\end{pmatrix}.$$

Hence the perpendicular part of $\vec{v}_3$ is

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}-\begin{pmatrix}2\\3\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix},$$

which also has norm 2. So the normalized basis is

$$\left\{\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\right\}.$$

4. *(15 pts.)* Let $\underset{\sim}{A}:\mathcal{V}\to\mathcal{U}$ be linear, where $\mathcal{V}$ and $\mathcal{U}$ are finite-dimensional vector spaces. Show (without extensive calculations) that there exist bases for $\mathcal{V}$ and $\mathcal{U}$ with respect to which $\underset{\sim}{A}$ is represented by a matrix of the type exemplified by

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}.$$

Namely, for $j\le p$ the j th column has a 1 in the j th row and zeros elsewhere, and for $j>p$ the j th column is entirely zero; here p can be any integer from 0 to $\min(\dim\mathcal{U},\dim\mathcal{V})$. *Hint:* Remember the proof of the rank-nullity theorem (dim ker + dim ran = dim dom);

it begins "Choose a basis for the kernel, and extend it to … ". (*Remark:* When $\mathcal{V} = \mathcal{U}$, the two bases involved here are probably not the same!)

Choose a basis for the kernel of $\underset{\sim}{A}$, and extend it to a basis for all of $\mathcal{V}$. Call the vectors (if any) in the extension $\{e_1, \ldots, e_p\}$, and put them *first* in the basis. Their images $\{\underset{\sim}{A}(e_1), \ldots, \underset{\sim}{A}(e_p)\}$ are linearly independent (else some linear combination of the e's would fall into the kernel) and they form a basis for the range of $\underset{\sim}{A}$. Extend this basis to a basis for all of $\mathcal{U}$ (putting the image vectors *first* in this basis). Then the first p elements of the $\mathcal{V}$ basis are mapped (respectively) into the first p elements of the $\mathcal{U}$ basis, and the remaining elements of the $\mathcal{V}$ basis are mapped into the zero vector. This is precisely the situation described by a matrix of the indicated type.

Alternative argument: It is clear that any row-reduced matrix can be transformed into a matrix of the indicated type by *column* reduction. But row reduction is equivalent to some change of basis in the codomain, and column reduction is equivalent to some change of basis in the domain. Therefore, some combination of two basis changes converts the matrix of $\underset{\sim}{A}$ to the indicated type.

5. *(15 pts.)* Let $\underset{\sim}{A} : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation with the matrix (with respect to the natural basis)

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 4 & 4 & 4 \end{pmatrix}.$$

(a) Find the range of $\underset{\sim}{A}$ by the Fredholm method. (Give a basis for the range.)

First find the kernel of $\underset{\sim}{A}^*$: Row reduction leads from $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 4 \\ 2 & 0 & 4 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus the kernel comprises the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying $\begin{aligned} x + 2z &= 0, \\ y + z &= 0. \end{aligned}$ So a basis for the kernel consists of the single vector $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. Therefore, the vectors in the range are those satisfying $-2x - y + z = 0$. Clearly this subspace is 2-dimensional; we can get a basis by, for instance, finding the solution with $x = 0$, $y = 1$ and the solution with $x = 1$, $y = 0$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Note that this solution could be read off from the nonzero rows of the reduced matrix (see Method 1 in (b)).

(b) Find the range of $\underset{\sim}{A}$ by a more direct method. (Of course, your answer should agree with (a).)

Method 1: The range is the span of the columns of A. To construct a basis, write the columns as rows and reduce; we already did this calculation in (a). Writing the nonzero rows of the reduced matrix as columns, we get the same answer as in (a).

Method 2: Reduce the matrix formed by augmenting $\underset{\sim}{A}$ (not $\underset{\sim}{A}^*$) by the final column $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The reduced matrix is $\begin{pmatrix} 1 & 0 & 2 & x \\ 0 & 1 & -1 & (y-2x)/4 \\ 0 & 0 & 0 & z-2x-y \end{pmatrix}$. Therefore, the vectors in the range are those that satisfy $-2x - y + z = 0$. This is the same equation we encountered and solved in (a).

6. *(20 pts.)* Define $\underset{\sim}{L}: \mathbf{R}^3 \to \mathbf{R}^2$ by $\underset{\sim}{L}(\vec{x}) = A\vec{x}$, $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$. (That is, A is the matrix of $\underset{\sim}{L}$ with respect to the natural bases.) Find the matrix representing $\underset{\sim}{L}$ with respect to the basis $\left\{ \vec{u}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ for $\mathbf{R}^2$ and the basis

$$\left\{ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

for $\mathbf{R}^3$.

Let $C = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (the matrices formed directly from the eigenvectors). Then $\underset{\sim}{C}$ maps $\vec{u}$-coordinates to natural coordinates (since $C\begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \alpha^1 \vec{u}_1 + \alpha^2 \hat{u}_2$), and $\underset{\sim}{B}$ maps $\vec{v}$-coordinates to natural coordinates. Thus the matrix we want is $M = C^{-1}AB$. Some calculation shows that $C^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ and hence that $M = \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

We can check this by the "k th-column rule":

$$\underset{\sim}{A}\,\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\vec{u}_1, \qquad \underset{\sim}{A}\,\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{u}_2 + \vec{u}_1, \qquad \underset{\sim}{A}\,\vec{v}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\hat{u}_2 - \vec{u}_1.$$

The coefficients match the columns of M, as they should.