Another proof of the spectral theorem

Reference: H. S. Wilf, "An algorithm-inspired proof of the spectral theorem in E^n", *Amer. Math. Monthly* **88**, 49–50 (1981).

This proof replaces some algebra by some topology (which may have been encountered in Math. 409, 436, etc.) I have filled in a few details; Wilf's article is very brief.

Wilf considers the real, symmetric case; he states, "The proof readily generalizes to the complex Hermitian case."

STEP 1: Prove the theorem in the 2×2 case:

Given $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, *there is an orthogonal matrix* $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ *such that* $R^{-1}AR$ *is diagonal.*

We take this to be known, since it is taught in [pre-]calculus courses as "rotation of axes for conic sections":

$$ax^2 + 2bxy + dy^2 \longrightarrow a'(x')^2 + d'(y')^2.$$

The formula for θ in terms of a, b, d will not concern us.

This is the step of the proof which requires a major change in the complex Hermitian case. The most general 2×2 Hermitian matrix involves 4 real parameters, not 3, and the most general 2×2 unitary matrix is described by 4 angles, not 1. One of these angles corresponds to an overall phase and can be ignored, leaving the 3 angular parameters of the group SU(2) of unitary matrices of determinant 1. We shall have to leave out the details, for lack of time.

STEP 2: Extrapolate this to the $n \times n$ case:

Given $A = \{A^j{}_k\}$, $A^j{}_k = A^k{}_j$ *real, and given a particular off-diagonal index pair* (j_0, k_0), $j_0 \neq k_0$, *there is an orthogonal* R *such that, if* $B \equiv R^{-1}AR$, *then* $B^{j_0}{}_{k_0} = 0 = B^{k_0}{}_{j_0}$.

Example: If $j_0 = 1$, $k_0 = 2$, take $R = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right)$, where the θ is that appropriate to the upper-left-hand 2×2 block of A. In general, take $R^{j_0}{}_{j_0} = \cos\theta$, $R^{j_0}{}_{k_0} = \sin\theta$, etc., for a suitable θ.

Observe that multiplying A on the right by R rotates the j_0, k_0 columns of A (within the subspace they span in $\mathbf{R}^n$) and leaves the other columns alone. The two numbers in the lth row of those two columns experience a rotation within $\mathbf{R}^2$. Similarly, multiplying on the left by $R^{-1} (= R^*)$ rotates the j_0, k_0 rows.

Remark: Iteration messes things up. Making $A^2{}_3 = 0$ makes $A^1{}_2 \neq 0$ once again. Consequently, more steps are necessary to complete the proof:

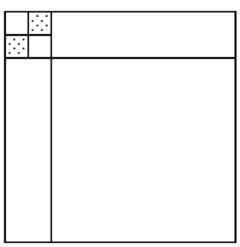
Step 3: Let us define $\mathrm{Od}(A)$ to be the sum of the squares of the off-diagonal elements of A:

$$\mathrm{Od}(A) \equiv \sum_{j \neq k} \left(A^j{}_k\right)^2 .$$

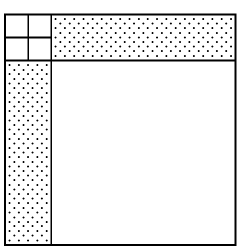
Note that $\mathrm{Od}(A) = 0$ iff A is diagonal; otherwise it is positive. Then note that in the situation of Step 2,

$$\begin{aligned} \mathrm{Od}(R^{-1}AR) &= \mathrm{Od}(A) - 2\left(A^{j_0}{}_{k_0}\right)^2 \\ &< \mathrm{Od}(A) \qquad (\text{if } A^{j_0}{}_{k_0} \neq 0). \end{aligned}$$

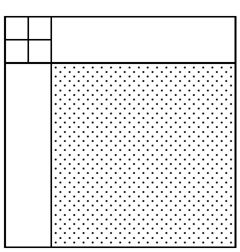
Proof: There are 3 kinds of off-diagonal elements:



1. $\{j,k\} = \{j_0, k_0\}$: These matrix elements were annihilated by the rotation; hence we subtract $2\left(A^{j_0}{}_{k_0}\right)^2$.



2. $\{j,k\}$ contains exactly one of $\{j_0, k_0\}$ (i.e., the row index or the column index, but not both, is one of the distinguished pair): These matrix elements were rotated among themselves, so the sum of their squares is unchanged.



3. $\{j,k\}$ does not involve $\{j_0, k_0\}$: These matrix elements were unchanged.

Step 4: Consider $\mathrm{Od}(R^{-1}AR) \equiv f(R)$ as a function of R. Note:

1) f is continuous (in the matrix elements of R).

2) The set of all orthogonal matrices R is a compact ($\iff$ closed and bounded, in this context) subset of $\mathbf{R}^{n^2}$, since it is defined by the equations

$$\sum_j \left(R^j{}_k\right)^2 = 1, \qquad \sum_j R^j{}_k R^j{}_l = 0.$$

Therefore, f has a *minimum value*, which is $f(R_0)$ for some R_0. This value must be zero (i.e., ${R_0}^{-1} A R_0$ is diagonal), since otherwise we could find a smaller value of $f(R)$ by Step 3. This proves the theorem.