§1.4. Stereographic projection and the point at infinity

In the preceding section we discussed sequences converging to a complex number. We now wish to formulate a notion of complex sequences tending (diverging) to ‘infinity’. Casting his mind back to the case of real sequences, the reader will recall two situations there which are typified by the following sequences: \( \{ n : n \in \mathbb{N} \} \) and \( \{ -n : n \in \mathbb{N} \} \). The first sequence tends to \(+\infty\), whereas the second tends to \(-\infty\), as \( n \to \infty \). (It is worthwhile to recall here that a sequence \( \{ a_n \} \) is said to tend to \(+\infty\) (resp. \(-\infty\)) if for very positive number \( T \), there is a positive integer \( N \), which may depend on \( T \), such that \( a_n > T \) (resp. \( a_n < -T \)) for every \( n \geq N \).) So any notion of a complex sequence tending to infinity must apply, in particular, to such real sequences. In addition, passing to the complex plane forces one to consider an infinite variety of sequences, all of which are equally worthy of diverging to infinity (e.g. \( \{ ni : n \in \mathbb{N} \}, \{ 1 - ni : n \in \mathbb{N} \}, \{ n + n^2 i : n \in \mathbb{N} \} \)). In order to motivate our actual definition, we consider the following.

Let \( \xi \), \( \eta \), and \( \zeta \) be mutually independent real variables, and let us consider the usual three-dimensional rectangular co-ordinate system, with the mutually orthogonal axes being denoted by the \( \xi \)-axis, the \( \eta \)-axis, and the \( \zeta \)-axis. Identify each complex number \( z = x + iy \) with the point \((x, y, 0)\) on the \( \xi \eta \)-plane; this provides a one-to-one correspondence between the said plane and \( \mathbb{C} \). Let \( \mathcal{R} \) denote the sphere of radius \( 1/2 \) centred at \((\xi, \eta, \zeta) = (0, 0, 1/2)\). This sphere, called the Riemann Sphere, sits atop the co-ordinate plane \( \zeta = 0 \), which is tangential to the sphere at the origin. The equation of \( \mathcal{R} \) is given by

\[
\xi^2 + \eta^2 + \left( \zeta - \frac{1}{2} \right)^2 = \frac{1}{4}.
\] (1.4.1)

Given any point \((x, y, 0)\) on the \( \xi \eta \)-plane, the straight line joining it to the north pole \((0, 0, 1)\) of \( \mathcal{R} \) intersects the Riemann sphere at precisely one point. On the other hand, given any point on the sphere other than \((0, 0, 1)\), the line joining it to the north pole can be extended to meet the \( \xi \eta \)-plane at exactly one point. Thus we get a one-to-one correspondence between \( \mathbb{C} \) (via its identification with the \( \xi \eta \)-plane) and \( \mathcal{R} \setminus \{(0, 0, 1)\} \). This identification procedure is termed stereographic projection. We think of the (extended) complex number ‘infinity’ as being the (extended) complex number corresponding to the north pole of the Riemann sphere. It is worth noting here that, in contrast to the real case where we deal with two infinities \((\pm \infty)\), in the complex case there is only one point at infinity, denoted by \( \infty \).

Suppose now that \( \{ z_n = x_n + iy_n : n \in \mathbb{N} \} \) is a sequence of complex numbers. For each positive integer \( n \), let \((\xi_n, \eta_n, \zeta_n)\) be the point on the Riemann sphere corresponding to \( z_n \). In view of the discussion in the previous paragraph, we say that the sequence \( \{ z_n \} \) diverges to infinity if the sequence \( \{(\xi_n, \eta_n, \zeta_n) : n \in \mathbb{N} \} \) approaches the north pole \((0, 0, 1)\), as \( n \) tends to infinity; that is, \( \xi_n, \eta_n \to 0 \) and \( \zeta_n \to 1^- \) as \( n \to \infty \). The upcoming (quantitative) analysis of the latter phenomenon will lead us to the formal definition of a complex sequence diverging to infinity.

As the point \((x_n, y_n, 0)\) lies on the straight line joining \((0, 0, 1)\) and \((\xi_n, \eta_n, \zeta_n)\), we obtain the equations

\[
\frac{x_n}{\xi_n} = \frac{y_n}{\eta_n} = \frac{1}{1 - \zeta_n}.
\]

(We assume for simplicity that neither \( \xi_n \) nor \( \eta_n \) is zero.) Therefore

\[
|z_n|^2 = x_n^2 + y_n^2 = \frac{\xi_n^2 + \eta_n^2}{(1 - \zeta_n)^2} = \frac{\zeta_n}{1 - \zeta_n},
\] (1.4.2)
the last equation stemming from the fact that the triple \((\xi_n, \eta_n, \zeta_n)\) satisfies the equation (1.4.1). Comparing the first and last terms in (1.4.2), we find that \(\lim_{n \to \infty} |z_n| = +\infty\) whenever \(\zeta_n \to 1^-\). On the other hand, elementary algebra provides the companion equations

\[
\xi_n = \frac{x_n}{1 + |z_n|^2}, \quad \eta_n = \frac{y_n}{1 + |z_n|^2}, \quad \text{and} \quad \zeta_n = \frac{|z_n|^2}{1 + |z_n|^2}.
\]

The last equation in (1.4.3) reveals that \(\zeta_n \to 1^-\) whenever \(|z_n| \to \infty\), whereas the first two equations, combined with the fact that \(\max\{|x_n|, |y_n|\} \leq |z_n|\) (Proposition 1.1.5), show that \(\xi_n\) and \(\eta_n\) converge to zero when \(|z_n| \to \infty\). Thus we see that the sequence \(\{(\xi_n, \eta_n, \zeta_n)\}\) approaches the north pole precisely when \(|z_n|\) tends to infinity, and this brings us to the following definition:

**Definition 1.4.1.** Suppose that \(\{z_n\}\) is a sequence of complex numbers. We say that \(\{z_n\}\) **diverges to infinity**, and write \(\lim_{n \to \infty} z_n = \infty\), if \(\lim_{n \to \infty} |z_n| = \infty\). Quantitatively this means the following: for every positive number \(T\), there is a positive integer \(N\), which may depend on \(T\), such that \(|z_n| > T\) for every \(n \geq N\).

**Remark 1.4.2.** (i) The reader will confirm that each of the five sequences mentioned in the first paragraph of this section diverges to infinity.

(ii) If \(\{a_n\}\) is a real sequence such that \(\lim_{n \to \infty} a_n = \pm \infty\), then \(\{a_n\}\), viewed as a complex sequence, diverges to infinity. Moreover, some of the subtler distinctions that obtain in the case of real sequences disappear when passing to the complex plane. For example, if

\[
b_n = \begin{cases} 
  n, & \text{if } n \text{ is odd;} \\
  -n, & \text{if } n \text{ is even,}
\end{cases}
\]

then, as a real sequence, the limit of \(\{b_n\}\) is neither \(+\infty\) nor \(-\infty\). Nonetheless, as a complex sequence, \(\{b_n\}\) does diverge to infinity.

(iii) Every complex sequence which diverges to infinity is unbounded. However, the following example demonstrates that the converse of this statement is false in general.

\[
z_n := \begin{cases} 
  i/n; & \text{if } n \text{ is odd;} \\
  ni; & \text{if } n \text{ is even.}
\end{cases}
\]

On the other hand, the reader will prove that a complex sequence is unbounded if and only if it has a subsequence which diverges to infinity.

(iv) In geometric terms, a sequence diverging to infinity is one which eventually escapes every disc centred at the origin.